# High energy bounds on wave operators

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### Abstract

In a general setting of scattering theory, we consider two self-adjoint operators $H_0$ and $H_1$ and investigate the behaviour of their wave operators $W_\pm(H_1, H_0)$ at asymptotic spectral values of $H_0$ and $H_1$. Specifically, we analyse when $\|(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}} P_0^{\mathrm{ac}}) f(H_0)\| < \infty$, where $P_j^{\mathrm{ac}}$ is the projector onto the subspace of absolutely continuous spectrum of $H_j$, and $f$ is an unbounded function ($f$-boundedness). We provide sufficient criteria both in the case of trace-class perturbations $V = H_1 - H_0$ and within the general setting of the smooth method of scattering theory, where the high-energy behaviour of the boundary values of the resolvent of $H_0$ plays a major role. In particular, we establish $f$-boundedness for the perturbed polyharmonic operator and for Schrödinger operators with matrix-valued potentials. Applications of these results include the problem of quantum backflow.

## 1 Introduction

The purpose of mathematical scattering theory is to compare two self-adjoint operators $H_0$, $H_1$ acting on a common Hilbert space $\mathcal{H}$ via their *wave operators* (or Møller operators), defined as the strong limits

$$W_\pm(H_1, H_0) := \operatorname*{s-lim}_{t \to \pm\infty} e^{-itH_1} e^{itH_0} P_0^{\mathrm{ac}}, \tag{1.1}$$

where $P_0^{\mathrm{ac}}$ is the projection onto the subspace $P_0^{\mathrm{ac}}\mathcal{H}$ of absolutely continuous spectrum of $H_0$. When these operators exist, they define isometries $P_0^{\mathrm{ac}}\mathcal{H} \to P_1^{\mathrm{ac}}\mathcal{H}$ that intertwine the absolutely continuous parts of $H_0$ and $H_1$ and can therefore be used to obtain information about $H_1$ based on information about $H_0$. This has many applications, in particular in quantum physics, where $H_0$ plays the role of a "free" Hamiltonian that can be investigated directly, and $H_1$ an "interacting", more complicated operator, that cannot be analysed directly. The wave operators are then used to characterise asymptotic properties of the dynamics given by the unitary group $e^{-itH_1}$ in terms of the "free" dynamics $e^{-itH_0}$.

Whereas many classical theorems in the field concentrate on establishing conditions on $H_0$ and $H_1$ that ensure existence of the wave operators $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$ (see the monographs [Yaf92, Yaf10, RS79] for a thorough presentation of the subject), we are interested in more quantitative questions at high spectral values of the operators. Namely, one would expect that in typical situations $W_\pm(H_1, H_0)$ approximates the identity on spectral subspaces for high spectral values.

One indication for this is as follows: Consider the *scattering operator* $S := W_+(H_1, H_0)^* W_-(H_1, H_0)$, which commutes with $H_0$ and hence in a diagonalization of $H_0$ acts by multiplication with an operator-valued function $\lambda \mapsto S(\lambda)$, where $\lambda$ are the spectral values of $H_0$. In relevant examples, $(S(\lambda) - \mathbf{1})$ is of Hilbert-Schmidt class ("finite total cross section" [ES80]) and its Hilbert-Schmidt norm decays as $|\lambda| \to \infty$ [Jen80, SY86].

In the present paper, we investigate a different and more direct question: we study the behaviour of $W_\pm(H_1, H_0) - \mathbf{1}$ at asymptotic spectral values of $H_1$ and $H_0$. For a flexible formalism, it turns out

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to be better to replace the identity $\mathbf{1}$ with the product of projections $P_1^{\mathrm{ac}} P_0^{\mathrm{ac}}$, and we define a pair $(H_1, H_0)$ of self-adjoint operators to be $f$-*bounded* (Def. 2.1) if their wave operators exist and satisfy

$$\|(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}} P_0^{\mathrm{ac}}) f(H_0)\| < \infty \tag{1.2}$$

for some unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$. Borrowing terminology from applications to Hamiltonians, we refer to bounds of the form (1.2) as "high energy bounds".

This question is partially motivated by our previous work on backflow, a surprising quantum mechanical effect which describes the situation where the probability current of a quantum particle in one dimension can flow in the direction opposite to its momentum. To quantify this effect in a situation with only asymptotic (in time) information on momentum distributions, bounds of the form (1.2) are essential [BCL17], where $H_0 = -\frac{d^2}{dx^2}$ is the one-dimensional Laplacian, $H_1 - H_0$ is a short range multiplication operator, and $f(\lambda) = \sqrt{\lambda}$.

More generally, high energy bounds are of interest whenever one wishes to quantify how small the effect of a perturbation $H_1 - H_0$ of a given self-adjoint operator $H_0$ on the unitary group $e^{-itH_0}$ is at asymptotic spectral values. This is not restricted to the particular one-dimensional Schrödinger operator setup encountered in backflow, but applies to a large variety of other situations. For example, one may think of pseudodifferential operators $H_0 = (-\frac{d^2}{dx^2} + m^2)^{1/2}$ to describe relativistic dynamics, integral operator potentials to describe noncommutative potential scattering [DG10, LV15], or applications to systems in higher dimensions or quantum field theory.

The aim of this article is to provide a unified analysis of $f$-bounds in an abstract setting, and to provide the tools for establishing them in a wide range of examples.

In Sec. 2, we recall some basic facts of scattering theory, and introduce the precise definition of $f$-boundedness as motivated above. We show with some a priori examples that $f$-boundedness depends crucially on the operator pair $(H_1, H_0)$: In some cases, it holds for all choices of $f$ (Example 2.5), whereas in other situations it holds for none (Example 2.4). Moreover, we show that under suitable assumptions, *mutual* $f$-boundedness introduces an equivalence relation on the set of all self-adjoint operators on $\mathcal{H}$ with purely absolutely continuous spectrum.

For treating examples closer to applications, there are two main methods of scattering theory, the *trace-class method* (where $H_1 - H_0$ is assumed to be of trace class) and the *smooth method* (where additional smoothness assumptions are put on the resolvents of $H_1$ and $H_0$), and we investigate $f$-boundedness in each of these.

We first briefly discuss the trace-class method in Sec. 3, giving sufficient conditions on a trace-class perturbation $V = H_1 - H_0$ for $f$-boundedness to hold (Prop. 3.2). Concretely, this applies in particular to rank-1 perturbations [Sim95].

Then, in Sec. 4, we discuss $f$-boundedness in the smooth method, where specifically the choice $f_\beta(\lambda) = (1 + \lambda^2)^{\beta/2}$, $\beta \in (0, 1)$, turns out to be advantageous. The main technical tool is to use the limiting absorption principle and study the wave operators in terms of boundary values of the resolvents of $H_0$ and $H_1$ at their spectra, taken in a suitable norm $\|\cdot\|_{\mathcal{X}, \mathcal{X}^*}$ defined by a Gelfand triple $\mathcal{X} \subset \mathcal{H} \subset \mathcal{X}^*$ (see, for example, [BA11] for a review of this technique). We pay particular attention to deriving sufficient conditions for (1.2) that can be expressed in terms of "free" data (referring to $H_0$, $R_0$, etc.) only, as required for applications. Our main result is that $f_\beta$-boundedness is essentially implied by the high-energy behaviour of the resolvent of $H_0$: if $R_0$ is locally Hölder continuous and $\|R_0(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} = O(|\lambda|^{-\beta})$, then $f_\beta$-boundedness follows (Theorem 4.9). This allows us, in particular, to treat the case where $H_0 = (-\Delta)^{\ell/2}$ is a fractional power of the Laplacian: For suitable $V$, we find that $f_\beta$-boundedness holds for $H_0$ and $H_0 + V$ whenever $0 < \beta \le 1 - \frac{1}{\ell}$ (Example 4.10), but this bound is sharp in general (Example 4.11).

We also ask whether this situation is stable under tensor product constructions, i.e., we consider $H_0 = H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$ where $H_A$ is of the type before, and $H_B$ has only point spectrum. Under certain conditions our results transfer to this situation (Corollary 4.14). In the case where $H_A$ is the negative Laplacian, in particular in applications to quantum physics, these $H_0$ are known as *matrix-valued Schrödinger operators* or *Schrödinger operators with matrix-valued potentials* (see, e.g., [GKM02, FLS07, KR08, CJLS16]), although we allow the matrices to become infinite-dimensional (Example 4.15). A particular problem occurs here for low-dimensional Laplacians if $H_B$ is of infinite rank; we discuss this in Example 4.16.

Finally, in Sec. 5, we return to our motivating backflow example from quantum mechanics, and show that semiboundedness of certain operators is preserved under perturbation with a wave operator.

Appendices A–C recall and supply some auxiliary results needed in the main text.

## 2   General setting

Throughout this article, our general setup will be the following. We consider two self-adjoint operators $H_0$ and $H_1$ on a common separable Hilbert space $\mathcal{H}$. In our notation, we use an index 0 or 1 to distinguish between the spectral resolutions and subspaces related to $H_0$ or $H_1$, e.g., $P_0^{\mathrm{ac}}$ is the projection onto the subspace of absolutely continuous spectrum of $H_0$, and $E_1$ is the spectral resolution of $H_1$. As the main quantity of interest, we consider the strong limits

$$W_\pm(H_1, H_0) := \operatorname*{s\text{-}lim}_{t \to \pm\infty} e^{-itH_1} e^{itH_0} P_0^{\mathrm{ac}} \tag{2.1}$$

and call them *wave operators* if they exist. We will use the following well-known results about wave operators (see, for example, [Yaf92, RS79]), and refer to them as (W1)–(W3).

(W1) $W_\pm(H_1, H_0)$ are partial isometries with initial space $\mathcal{H}_0^{\mathrm{ac}} = P_0^{\mathrm{ac}}\mathcal{H}$ and final space contained in $\mathcal{H}_1^{\mathrm{ac}}$. In case their final spaces coincide with $\mathcal{H}_1^{\mathrm{ac}}$, the wave operators are called *complete*. This is equivalent to the existence of $W_\pm(H_0, H_1)$, and implies $W_\pm(H_1, H_0)^* = W_\pm(H_0, H_1)$.

(W2) A *chain rule* holds: If $H_0, H_1, H_2$ are self-adjoint such that $W_\pm(H_1, H_0)$ and $W_\pm(H_2, H_1)$ exist, then also $W_\pm(H_2, H_0)$ exists, and

$$W_\pm(H_2, H_0) = W_\pm(H_2, H_1)W_\pm(H_1, H_0). \tag{2.2}$$

(W3) The *intertwiner property* holds: For any bounded Borel function $\varphi : \mathbb{R} \to \mathbb{C}$, one has

$$\varphi(H_1)W_\pm(H_1, H_0) = W_\pm(H_1, H_0)\varphi(H_0). \tag{2.3}$$

In many situations, $W_\pm(H_1, H_0)$ approximates the identity, or rather the operator $P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}$ in the presence of point spectrum, when restricted to spectral subspaces of $H_0$ for large spectral values. More specifically, one may find that $(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})f(H_0)$ is bounded despite the function $f$ being unbounded on the spectrum of $H_0$. This motivates the following definition; here and throughout the paper, $C(\mathbb{R})$ denotes the space of continuous real-valued functions on $\mathbb{R}$.

**Definition 2.1.** *Let $H_0$ and $H_1$ be two self-adjoint operators on a Hilbert space $\mathcal{H}$ such that their wave operators $W_\pm(H_1, H_0)$ exist, and let $f \in C(\mathbb{R})$. Then the pair $(H_1, H_0)$ is called $f$-bounded if $(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})f(H_0)$ is bounded.*

*If both $(H_1, H_0)$ and $(H_0, H_1)$ are $f$-bounded, then we call the operators mutually $f$-bounded, denoted $H_1 \sim_f H_0$.*

Clearly one is interested here in functions $f$ that grow as $\lambda \to \pm\infty$; a typical choice would be $f_\beta(\lambda) := (1 + \lambda^2)^{\beta/2}$, $\beta > 0$. This raises the question which rate of growth of $f(\lambda)$ as $\lambda \to \pm\infty$ is still compatible with $f$-boundedness, to be investigated in later sections.

Heuristically, $f$-boundedness should be determined by the behaviour of the wave operator at large spectral values of $H_0$ *and* $H_1$. In the following lemma, we show that under the additional restriction $f(H_1) - f(H_0) \in \mathfrak{B}(\mathcal{H})$, this can in fact be made precise. This condition holds for a large class of $f$ in case $H_1 - H_0$ is bounded, see Lemma B.1 in Appendix B.

**Lemma 2.2.** *Let $H_0$, $H_1$ be two self-adjoint operators, and $f \in C(\mathbb{R})$. Suppose that $W := W_\pm(H_1, H_0)$ exists and is complete, and that $f(H_1) - f(H_0)$ is bounded. Then the following statements are equivalent:*

*(i) $\left(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\right)f(H_0) \in \mathfrak{B}(\mathcal{H})$;*

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*(ii)* $E_1(-\lambda,\lambda)^\perp\big(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f(H_0)E_0(-\lambda,\lambda)^\perp \in \mathfrak{B}(\mathcal{H})$ *for some* $\lambda > 0$;

*(iii)* $E_1(-\lambda,\lambda)^\perp\big(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f(H_0)E_0(-\lambda,\lambda)^\perp \in \mathfrak{B}(\mathcal{H})$ *for all* $\lambda > 0$.

*Here* $E_j(-\lambda,\lambda)^\perp := \mathbf{1} - E_j(-\lambda,\lambda)$.

*Proof.* It is clear that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (i), since $f(H_0)E_0(-\lambda,\lambda)$ is bounded, it only remains to show that $E_1(-\lambda,\lambda)\big(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f(H_0) \in \mathfrak{B}(\mathcal{H})$. To that end, we consider a compact set $\Delta \subset \mathbb{R}$ and define the bounded Borel functions $f_\Delta := f \cdot \chi_\Delta$. Using the intertwining property (W3) of $W$, we get

$$\begin{aligned}
E_1(-\lambda,\lambda)\big(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f_\Delta(H_0) = \\
E_1(-\lambda,\lambda)f_\Delta(H_1)(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}) + E_1(-\lambda,\lambda)P_1^{\mathrm{ac}}\big(f_\Delta(H_1) - f_\Delta(H_0)\big)P_0^{\mathrm{ac}}.
\end{aligned} \tag{2.4}$$

As $\Delta \to \mathbb{R}$, this operator remains bounded by assumption. $\qquad\square$

It is interesting to note that *mutual $f$-boundedness* of two operators is clearly symmetric and reflexive; and under slight extra assumptions, it is also transitive.

**Lemma 2.3.** *Suppose that $H_0, H_1, H_2$ are three self-adjoint operators with $H_0 \sim_f H_1$ and $H_1 \sim_f H_2$ for some $f \in C(\mathbb{R})$, and such that $(\mathbf{1} - P_1^{\mathrm{ac}})f(H_0)$ and $(\mathbf{1} - P_1^{\mathrm{ac}})f(H_2)$ are bounded. Then $H_0 \sim_f H_2$.*

Thus, in particular, $\sim_f$ is an equivalence relation on the set of all self-adjoint operators on $\mathcal{H}$ that have purely absolutely continuous spectrum.

*Proof.* Using the chain rule (W2), we see that $W_\pm(H_2, H_0)$ exists. Now we cut down $f$ to $f_\Delta = f \cdot \chi_\Delta$ with a compact $\Delta \subset \mathbb{R}$. The intertwining property (W3) then yields

$$\begin{aligned}
\big(W_\pm(H_2, H_0) - P_2^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f_\Delta(H_0) = {}& \big(W_\pm(H_2, H_1) - P_2^{\mathrm{ac}}P_1^{\mathrm{ac}}\big)f_\Delta(H_1)W_\pm(H_1, H_0) \\
& + P_2^{\mathrm{ac}}\big(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}\big)f_\Delta(H_0) \\
& + P_2^{\mathrm{ac}}(P_1^{\mathrm{ac}} - \mathbf{1})P_0^{\mathrm{ac}}f_\Delta(H_0).
\end{aligned} \tag{2.5}$$

As $\Delta \to \mathbb{R}$, the first and second terms are bounded since $H_2 \sim_f H_1$ and $H_1 \sim_f H_0$, respectively, and the third term is bounded by our extra assumption. Since our assumptions are symmetric in $H_2$ and $H_0$, this finishes the proof. $\qquad\square$

As an aside, we also mention that in the situation of two mutually $f$-bounded operators $H_0$, $H_1$, one has $W_\pm(H_0, H_1)^* = W_\pm(H_1, H_0)$ by (W1), with which it is easy to show that $P_1^{\mathrm{ac}}(f(H_1) - f(H_0))P_0^{\mathrm{ac}}$ is bounded. This shows that the assumption used in Lemma 2.2 is quite natural.

A pair of self-adjoint operators $H_0$, $H_1$ that have wave operators can exhibit very different behaviour regarding $f$-boundedness, as we now demonstrate with two examples: The first example shows that $f$-boundedness can fail for all unbounded $f$, whereas the second example shows that $f$-boundedness can hold for every $f$.

**Example 2.4.** *Let $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $H_0 := -i\frac{d}{dx}$. Let $v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be non-zero and real, and define $H_1 := H_0 + V$, where $V$ is the operator multiplying with $v$. Then the wave operators $W_\pm := W_\pm(H_0, H_1)$ and $W_\pm(H_1, H_0)$ exist, and $(H_1, H_0)$ is $f$-bounded if and only if the function $f$ is bounded.*

*Proof.* Note that $H_0$ and $H_1$ are self-adjoint on their natural domains. As shown in [Yaf92, p. 83-84], $H_0$ and $H_1$ are unitarily equivalent and have absolutely continuous simple spectrum covering the full real axis. Furthermore, the wave operators $W_\pm := W_\pm(H_0, H_1)$ exist and are unitary. Explicitly, they act as

$$(W_\pm\psi)(x) = w_\pm(x)\psi(x), \qquad w_\pm(x) := \exp i\int_x^{\pm\infty} v(y)\,dy. \tag{2.6}$$

This implies in particular that the wave operators are complete, and $W_\pm(H_1, H_0) = W_\pm^*$ acts by multiplication with $\overline{w_\pm}$.

As multiplication operators, $W_\pm - 1$ commute with the unitaries $(U(p)\psi)(x) := e^{ipx}\psi(x)$, whereas $U(p)H_0 U(p)^{-1} = H_0 - p$. Thus $(W_\pm^* - 1)f(H_0)$ is bounded if and only if

$$U(p)(W_\pm^* - 1)f(H_0)U(p)^{-1} = (W_\pm^* - 1)f(H_0 - p) = (f(H_0 - p)(W_\pm - 1))^* \qquad (2.7)$$

is bounded uniformly in $p \in \mathbb{R}$. But since $W_\pm - 1 \neq 0$, we can choose a nonzero vector $\psi$ in its image, and then $\|f(H_0 - p)\psi\|$ is not uniformly bounded in $p$, unless $f$ is bounded as a function. $\qquad\square$

**Example 2.5.** *Let $H_0$ be a self-adjoint operator with purely absolutely continuous spectrum, $V$ a self-adjoint bounded operator on the same Hilbert space $\mathcal{H}$, and $H_1 := H_0 + V$. Assume that the wave operators $W_\pm(H_1, H_0)$ exist and that there exists a compact $\Delta \subset \sigma(H_0)$ such that $V\mathcal{H} \subset E_0(\Delta)\mathcal{H}$. Then $(H_1, H_0)$ is $f$-bounded for every $f \in C(\mathbb{R})$.*

*Proof.* With $E_0^\perp := 1 - E_0(\Delta)$, our assumption can be rephrased as $0 = VE_0^\perp = (H_1 - H_0)E_0^\perp$, which implies $H_1 E_0^\perp = H_0 E_0^\perp = E_0^\perp H_0$ on $\mathrm{dom}\, H_0 = \mathrm{dom}\, H_1$, and, by self-adjointness, also $E_0^\perp H_1 = E_0^\perp H_0$ on this domain. A power series calculation on analytic vectors for $H_0$ then gives $E_0^\perp e^{-itH_0} = e^{-itH_1}E_0^\perp$, and therefore

$$W_\pm(H_1, H_0)E_0^\perp = \operatorname*{s-lim}_{t \to \pm\infty} e^{-itH_1}e^{itH_0}E_0^\perp = \operatorname*{s-lim}_{t \to \pm\infty} e^{-itH_1}E_0^\perp e^{itH_0} = E_0^\perp. \qquad (2.8)$$

Thus $(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}}) = P_1^{\mathrm{ac}}(W_\pm(H_1, H_0) - 1) = P_1^{\mathrm{ac}}(W_\pm(H_1, H_0) - 1)E_0(\Delta)$, and for arbitrary $f \in C(\mathbb{R})$, we have the bound

$$\|(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}})f(H_0)\| = \|(W_\pm(H_1, H_0) - P_1^{\mathrm{ac}})f_\Delta(H_0)\| \leq 2\|f_\Delta\|_\infty < \infty. \qquad (2.9)$$

$\qquad\square$

A concrete realisation of this situation on $\mathcal{H} = L^2(\mathbb{R}, dx)$ is given by $H_0 = -\frac{d^2}{dx^2}$ and $V$ an integral operator such that the Fourier transform of its kernel lies in $C_0^\infty(\mathbb{R} \times \mathbb{R})$. Then $V$ is trace-class, which implies existence and completeness of the wave operators by the Kato-Rosenblum theorem [RS79, Thm. XI.8], and the assumption $V\mathcal{H} \subset E_0(\Delta)\mathcal{H}$ follows from the support of the integral kernel.

## 3  Trace class perturbations and $f$-boundedness

The Kato-Rosenblum theorem states that if $H_0$ and $H_1$ are self-adjoint and their difference is trace class, then $W_\pm(H_1, H_0)$ exist and are complete. Examples of such trace class perturbations are given by integral operators with suitable kernels, or rank one perturbations as the simplest case.

We now investigate $f$-boundedness in this setting and first recall some relevant notions. For a self-adjoint operator $H_0$ on $\mathcal{H}$ and a vector $\xi \in P_0^{\mathrm{ac}}\mathcal{H}$, we define

$$\|\xi\|_{H_0}^2 := \operatorname*{ess\,sup}_{\lambda \in \hat\sigma(H_0)} \left| \frac{d\langle \xi, E_0(\lambda)\xi \rangle}{d\lambda} \right| = \operatorname*{ess\,sup}_{\lambda \in \hat\sigma(H_0)} \|\xi_\lambda\|_\lambda^2 \in [0, +\infty],$$

where $\hat\sigma(H_0)$ is the core of the spectrum $\sigma(H_0)$. The second expression refers to the direct integral decomposition of the absolutely continuous subspace,

$$P_0^{\mathrm{ac}}\mathcal{H} = \int_{\hat\sigma(H_0)}^\oplus \mathfrak{h}(\lambda)d\lambda, \qquad (3.1)$$

namely $\xi_\lambda$ is the component of $\xi \in P_0^{\mathrm{ac}}\mathcal{H}$ in $\mathfrak{h}(\lambda)$, and $\| \cdot \|_\lambda$ is the norm of $\mathfrak{h}(\lambda)$ [Yaf92, p. 32]. The set of all $\xi$ with finite $\|\xi\|_{H_0}$ is $\| \cdot \|$-dense in $P_0^{\mathrm{ac}}\mathcal{H}$, and $\| \cdot \|_{H_0}$ is a norm on it [RS79].

Note that if $H_0, H_1$ are two self-adjoint operators with complete wave operators, then $W_\pm(H_0, H_1) : P_1^{\mathrm{ac}}\mathcal{H} \to P_0^{\mathrm{ac}}\mathcal{H}$ are unitaries intertwining the absolutely continuous parts of $H_0$ and $H_1$. This implies that we can identify the direct integral decompositions of $P_0^{\mathrm{ac}}\mathcal{H}$ and $P_1^{\mathrm{ac}}\mathcal{H}$, and $\|(W_\pm(H_0, H_1)\xi)_\lambda\|_\lambda = \|\xi_\lambda\|_\lambda$ for each $\lambda \in \hat\sigma(H_0)$. In particular, $\|W_\pm(H_0, H_1)\xi\|_{H_0} = \|\xi\|_{H_1}$ in this situation.

The following lemma due to Rosenblum [RS79, Lemma 1, p.23] will be essential in the following.

**Lemma 3.1.** *Let $H_0$ be a self-adjoint on $\mathcal{H}$ and $\psi, \xi$ vectors in $\mathcal{H}$, with $\xi \in P_0^{\mathrm{ac}}\mathcal{H}$ such that $\|\xi\|_{H_0} < \infty$. Then*

$$\int_{-\infty}^{\infty} |\langle \psi, e^{\pm itH_0}\xi\rangle|^2 dt \leq 2\pi \|\xi\|_{H_0}^2 \|\psi\|^2 < \infty. \tag{3.2}$$

As in the Kato-Rosenblum theorem, we now consider two self-adjoint operators $H_0, H_1$ such that $V := H_1 - H_0$ is trace class, and hence has an expansion $V\psi = \sum_n t_n\langle \xi_n, \psi\rangle \xi_n$, $\psi \in \mathcal{H}$, where the $\xi_n$ form an orthonormal basis of $\mathcal{H}$ and the sequence $(t_n)$ is summable, $\sum_n |t_n| < \infty$.

The idea of the $f$-boundedness result presented below is to choose $V := H_1 - H_0$ such that $Vf(H_0)$ also

$$\|Vf(H_0)\|_{H_1,H_0}^{\Lambda} := \sum_n |t_n| \|P_1^{\mathrm{ac}} E_1(-\Lambda, \Lambda)^{\perp}\xi_n\|_{H_1} \|P_0^{\mathrm{ac}} E_0(-\Lambda, \Lambda)^{\perp}f(H_0)\xi_n\|_{H_0}, \tag{3.3}$$

is finite, where $\Lambda \geq 0$ is arbitrary.

**Proposition 3.2.** *Let $H_0, H_1$ be self-adjoint with $V := H_1 - H_0$ of trace class and $\|Vf(H_0)\|_{H_1,H_0}^{\Lambda} < \infty$ for some $\Lambda \geq 0$. Let $f \in C(\mathbb{R})$ such that $f(H_1) - f(H_0)$ is bounded. Then $W_{\pm}(H_1, H_0)$ exist, are complete, and satisfy*

$$\|(W_{\pm}(H_1, H_0) - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})f(H_0)\| \leq 2\pi \|Vf(H_0)\|_{H_1,H_0}^{\Lambda} < \infty.$$

*In particular, $(H_1, H_0)$ is $f$-bounded.*

*Proof.* Existence and completeness of $W := W_{\pm}(H_1, H_0)$ follow from the Kato-Rosenblum Theorem. To obtain $f$-boundedness, we first recall that for $\varphi \in \mathrm{dom}H_1$, $\psi \in \mathrm{dom}H_0$, we have

$$\frac{d}{dt}\langle \varphi, P_1^{\mathrm{ac}} e^{\mp itH_1} e^{\pm itH_0} P_0^{\mathrm{ac}}\psi\rangle = \mp i\langle \varphi, P_1^{\mathrm{ac}} e^{\mp itH_1} V e^{\pm itH_0} P_0^{\mathrm{ac}}\psi\rangle. \tag{3.4}$$

After integration, this yields

$$\langle \varphi, (W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})\psi\rangle = \pm i \int_0^{\infty} \langle P_1^{\mathrm{ac}}\varphi, e^{\mp itH_1} V e^{\pm itH_0} P_0^{\mathrm{ac}}\psi\rangle dt. \tag{3.5}$$

The proof is based on this identity, the expansion $V\psi = \sum_n t_n\langle \xi_n, \psi\rangle \xi_n$ of $V$, and Lemma 3.1. Let $\varphi, \psi \in \mathrm{dom}H_0$. Then, with $E_j^{\Lambda} := E_j(-\Lambda, \Lambda)^{\perp}$,

$$\left|\langle E_1^{\Lambda}\varphi, (W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})f(H_0)E_0^{\Lambda}\psi\rangle\right| = \left|\int_0^{\infty} \langle \varphi, e^{\mp itH_1} P_1^{\mathrm{ac}} E_1^{\Lambda} V E_0^{\Lambda} e^{\pm itH_0} f(H_0) P_0^{\mathrm{ac}}\psi\rangle dt\right|$$

$$\leq \sum_n |t_n| \left|\int_0^{\infty} \langle \varphi, e^{\mp itH_1} P_1^{\mathrm{ac}} E_1^{\Lambda}\xi_n\rangle\langle E_0^{\Lambda}\xi_n, e^{\pm itH_0} f(H_0) P_0^{\mathrm{ac}}\psi\rangle dt\right|$$

$$\leq \sum_n |t_n| \left(\int_0^{\infty} |\langle P_1^{\mathrm{ac}} E_1^{\Lambda}\xi_n, e^{\pm itH_1}\varphi\rangle|^2 dt \cdot \int_0^{\infty} |\langle f(H_0) P_0^{\mathrm{ac}} E_0^{\Lambda}\xi_n, e^{\pm itH_0} P_0^{\mathrm{ac}}\psi\rangle|^2 dt\right)^{1/2}.$$

We can now use Lemma 3.1 to estimate the integrals, and arrive at the bound

$$\left|\langle \varphi, E_1^{\Lambda}(W - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})f(H_0)E_0^{\Lambda}\psi\rangle\right| \leq 2\pi \sum_n |t_n| \|P_1^{\mathrm{ac}} E_1^{\Lambda}\xi_n\|_{H_1} \|f(H_0)E_0^{\Lambda}P_0^{\mathrm{ac}}\xi_n\|_{H_0} \|\varphi\|\|\psi\|$$

$$= 2\pi \|Vf(H_0)\|_{H_1,H_0}^{\Lambda} \|\varphi\|\|\psi\|.$$

In view of Lemma 2.2, this finishes the proof. $\qquad\square$

We note that the assumption $f(H_1) - f(H_0)$ being bounded was only used for the reference to Lemma 2.2 and can therefore be dropped for $\Lambda = 0$.

Particular examples of trace-class perturbations that have attracted continued attention are perturbations by rank one operators $V = \langle \xi, \cdot \rangle \xi$ with some $\xi \in \mathcal{H}$ [Sim95]. In that case, the bound from the previous proposition is $2\pi \|P_1^{\mathrm{ac}} E_1(-\Lambda, \Lambda)^{\perp}\xi\|_{H_1} \|P_0^{\mathrm{ac}} E_0(-\Lambda, \Lambda)^{\perp}f(H_0)\xi\|_{H_0}$, and in order to have

it finite, we need to control the "spectral norms" of both $H_0$ and $H_1$. Whereas the norm coming from $H_0$ can typically be controlled directly in applications, this is typically not the case for $H_1$. Let us therefore clarify how an estimate on $\|P_1^{\mathrm{ac}} E_1(-\Lambda,\Lambda)^{\perp}\xi\|_{H_1} = \operatorname{ess\,sup}_{|\lambda|\geq\Lambda}\left|\frac{d\langle\xi,E_1(\lambda)\xi\rangle}{d\lambda}\right|$ can be given in terms of $H_0$:

With the resolvents $R_0, R_1$ of $H_0, H_1$, one has

$$\frac{d\langle\xi,E_j(\lambda)\xi\rangle}{d\lambda} = \frac{1}{\pi}\operatorname{Im}\langle\xi,R_j(\lambda+i0)\xi\rangle, \tag{3.6}$$

and for a rank one perturbation $H_1 = H_0 + \langle\xi,\cdot\,\rangle\xi$, one moreover has the Aronszajn-Krein formula [Sim95]

$$\langle\xi,R_1(\lambda+i0)\xi\rangle = \frac{\langle\xi,R_0(\lambda+i0)\xi\rangle}{1+\langle\xi,R_0(\lambda+i0)\xi\rangle}. \tag{3.7}$$

It therefore follows that in case $\xi$ is such that $\|\xi\|_{H_0} < \infty$ and the boundary values $\langle\xi,R_0(\lambda+i0)\xi\rangle$ converge to 0 as $|\lambda|\to\infty$, then also $\|\xi\|_{H_1} < \infty$.

As a concrete example, we may take $\mathcal{H} = L^2(\mathbb{R},dx)$ with $H_0 = -\frac{d^2}{dx^2}$ and $H_1 = H_0 + \langle\xi,\cdot\,\rangle\xi$, where $\xi \in \mathscr{S}(\mathbb{R})$ is a Schwartz function. Then the spectral measure of $P = i\frac{d}{dx}$ is given by $d\langle\xi,E_P(p)\xi\rangle = |\tilde{\xi}(p)|^2 dp$ with $\tilde{\xi}$ the Fourier transform of $\xi \in \mathcal{H}$. After substituting $\lambda = p^2$, this shows that the spectral measure of $H_0 = P^2$ is $d\langle\xi,E_0(\lambda)\xi\rangle = \frac{1}{2\sqrt{\lambda}}\left(|\tilde{\xi}(\sqrt{\lambda})|^2 + |\tilde{\xi}(-\sqrt{\lambda})|^2\right)d\lambda$. Hence $\|E_0(-\Lambda,\Lambda)^{\perp}\xi\|_{H_0} < \infty$ and $\|E_0(-\Lambda,\Lambda)^{\perp}f(H_0)\xi\|_{H_0} < \infty$ for any $\xi \in \mathscr{S}(\mathbb{R})$, $\Lambda > 0$ and polynomially bounded $f \in C(\mathbb{R})$. It is also well known that $\langle\xi,R_0(\lambda\pm i0)\xi\rangle \to 0$ as $|\lambda|\to\infty$ (see Example 4.10). Hence $(H_1,H_0)$ is $f$-bounded in this situation.

# 4 Smooth method and $f$-boundedness

We now discuss another specific setting of scattering theory, known as the *smooth method*, which is applicable in particular to cases where one of the operators $H_j$ is a (pseudo)differential operator.

The idea behind this is as follows. If $H$ is a self-adjoint operator and $R(z) = (H-z)^{-1}$ its resolvents, then the operator-valued function $z \mapsto R(z)$ is certainly analytic on the half planes $\mathbb{H}_{\pm} := \mathbb{R} \pm i\mathbb{R}_+$. One now demands that, in a suitable topology, it extends to the boundaries of the half planes, and that the extended functions are locally Hölder continuous on $\overline{\mathbb{H}}_{\pm}$ (possibly with the exception of a null set). In this case $H$ is called *smooth*. If both $H_0$ and $H_1$ are smooth, then the wave operators $W_{\pm}(H_1,H_0)$ automatically exist and are complete. In fact, one can express the wave operators, as well as other relevant quantities, in terms of the boundary values of the resolvents, $R_1(\lambda\pm i0)$ and $R_0(\lambda\pm i0)$. We recall the basic facts about this setting in Sec. 4.1, mainly in the spirit of [KK71, BA11]; see also [Yaf92].

In the context of this setting, we are interested in mutual $f$-boundedness of two operators $H_1$ and $H_0$, where we restrict to $f(\lambda) = (1+\lambda^2)^{\beta/2}$ with some $\beta \in (0,1)$. It turns out that this is implied by the behaviour of $R_0(\lambda\pm i0)$ at large $\lambda$ alone: If a certain norm of this operator is $O(|\lambda|^{-\beta})$, then $H_0 \sim_f H_1$ follows (Theorem 4.9).

We apply this result to examples of pseudodifferential operators (Sec. 4.3) and investigate stability under tensor product constructions (Sec. 4.4).

## 4.1 Setting of the smooth method

Let $\mathcal{X}$ be a Banach space[1] which is continuously and densely embedded in $\mathcal{H}$. The scalar product $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathcal{H}$ then yields an embedding $\mathcal{H} \subset \mathcal{X}^*$, $\varphi \mapsto \langle\,\cdot\,,\varphi\rangle$, where $\mathcal{X}^*$ denotes the conjugate dual of $\mathcal{X}$, yielding a so-called Gelfand triple $\mathcal{X} \subset \mathcal{H} \subset \mathcal{X}^*$. We assume that the embedding $\mathcal{H} \subset \mathcal{X}^*$ is dense, so that $\langle\,\cdot\,,\cdot\,\rangle$ extends to a dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$, which we denote by the same symbol. In this setting, let us define the class of operators $H$ of interest.

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[1]In most applications, $\mathcal{X}$ is actually a Hilbert space, but we stress that in this case, the dual pairing $\langle\,\cdot\,,\cdot\,\rangle$ is normally *not* the scalar product on $\mathcal{X}$; the latter plays no role in our investigation.

**Definition 4.1.** *Let $H$ be a self-adjoint operator on a dense domain in $\mathcal{H}$ and $R(z)$ its resolvents. We call $H$ an $\mathcal{X}$-smooth operator if there exists an open set $U \subset \mathbb{R}$ of full (Lebesgue) measure such that the limits*

$$R(\lambda \pm i0) := \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon), \quad \lambda \in U, \tag{4.1}$$

*exist in $\mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$, and the extended functions $R : \mathbb{H}_\pm \cup U \to \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ are locally Hölder continuous.*

In this situation, it follows that $U \subset \sigma_{\mathrm{ac}}(H)$ and for any Borel set $\Delta \subset U$, one has $E(\Delta)\mathcal{H} \subset P^{\mathrm{ac}}\mathcal{H}$. The locally Hölder continuous map $A : U \to \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ given by

$$A(\lambda) = \frac{1}{2\pi i}\big(R(\lambda + i0) - R(\lambda - i0)\big) \tag{4.2}$$

equals the weak derivative $\frac{dE}{d\lambda}$ where $E$ are the spectral projections of $H$, i.e.,

$$\frac{d}{d\lambda}\langle \varphi, E(\lambda)\psi \rangle = \langle \varphi, A(\lambda)\psi \rangle \quad \text{for all } \varphi, \psi \in \mathcal{X}, \lambda \in U. \tag{4.3}$$

It follows that $A(\lambda)$ diagonalizes the absolutely continuous part of $H$, in the sense that for bounded Borel functions $f$,

$$\langle \varphi, f(H)P^{\mathrm{ac}}\psi \rangle = \int d\lambda\, f(\lambda)\,\langle \varphi, A(\lambda)\psi \rangle \quad \text{for all } \varphi, \psi \in \mathcal{X}. \tag{4.4}$$

Also, we note that the $A(\lambda)$ are positive as quadratic forms on $\mathcal{X} \times \mathcal{X}$, and therefore one has

$$\forall \varphi \in \mathcal{X}: \quad \|A(\lambda)\varphi\|_{\mathcal{X}^*}^2 \leq \|A(\lambda)\|_{\mathcal{X},\mathcal{X}^*}\langle \varphi, A(\lambda)\varphi \rangle. \tag{4.5}$$

Equally, one can start with the maps $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ and deduce the properties of the resolvents $R(\lambda \pm i0)$, see [BA11, Sec. 3]. We give the following sufficient criterion:

**Lemma 4.2.** *Let $A(\lambda) = \frac{dE}{d\lambda} \in \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ be locally Hölder continuous in $\lambda \in U$, where $U \subset \mathbb{R}$ is an open set of full measure, and suppose that $\lambda \mapsto (1 + |\lambda|)^{-1}\|A(\lambda)\|_{\mathcal{X},\mathcal{X}^*}$ is integrable over $\mathbb{R}$. Then $H$ is $\mathcal{X}$-smooth, and (4.2) holds.*

To see this, one considers for $\mathrm{Im}\, z \neq 0$ the integral $R(z) = \int \frac{A(\lambda)}{\lambda - z}d\lambda$ weakly on $\mathcal{X} \times \mathcal{X}$, splits the integration region into a small interval $J$ around $\mathrm{Re}\, z \in U$ and into its complement, then applies the Privalov-Korn theorem (Lemma A.1) to the integral over $J$, and the integrability condition on $\mathbb{R} \backslash J$. See [BA11, Thm. 3.6] for details. We will give a quantitative version of this result below, in Proposition 4.7.

Passing to the setting of scattering theory, let $\Gamma_2(\mathcal{X}^*, \mathcal{X}) \subset \mathfrak{B}(\mathcal{X}^*, \mathcal{X})$ be the space of bounded operators that "factor through a Hilbert space"; that is, $V \in \Gamma_2(\mathcal{X}^*, \mathcal{X})$ is of the form $V = V_1^* V_0$ where $V_0, V_1 \in \mathfrak{B}(\mathcal{X}^*, \mathcal{K})$ with a Hilbert space $\mathcal{K}$.[2]

**Definition 4.3.** *A smooth scattering system $(H_0, H_1, \mathcal{X}, \mathcal{H})$ consists of two self-adjoint operators $H_0$ and $H_1$ on a common dense domain in the Hilbert space $\mathcal{H}$ which are both $\mathcal{X}$-smooth, such that $V := H_1 - H_0 \in \Gamma_2(\mathcal{X}^*, \mathcal{X})$.*

We can (and will) assume in this situation that both $H_j$ are smooth with respect to the same set $U$ of full measure. In practical examples, $\mathcal{X}$-smoothness of one operator (say, $H_0$) can usually be verified directly, whereas the $\mathcal{X}$-smoothness of $H_1$ is obtained by perturbation arguments. We give a well-known type of sufficient criterion (cf. [Yaf92, BA11]), and sketch its proof in our context.

**Lemma 4.4.** *Let $H_0$ be a self-adjoint operator on a dense set $\mathcal{D} \subset \mathcal{H}$ and $V = V^* \in \mathfrak{B}(\mathcal{X}^*, \mathcal{X})$. Suppose that $H_0$ is $\mathcal{X}$-smooth and that[3] $R_0(z) \in \mathrm{FA}(\mathcal{X}, \mathcal{X}^*)$ for all $z \in \mathbb{C} \backslash \mathbb{R}$. Then, $H_1 := H_0 + V$ is $\mathcal{X}$-smooth.*

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[2]If $\mathcal{X}$ is actually a Hilbert space, then $\Gamma_2(\mathcal{X}^*, \mathcal{X}) = \mathfrak{B}(\mathcal{X}^*, \mathcal{X})$; but in general, the inclusion may be proper [Pis86].

[3]$\mathrm{FA}(\mathcal{X}, \mathcal{X}^*)$ denotes the norm closure of the space of finite rank operators from $\mathcal{X}$ to $\mathcal{X}^*$. If $\mathcal{X}$ is a Hilbert space, $\mathrm{FA}(\mathcal{X}, \mathcal{X}^*)$ equals the space of compact operators.

*Proof.* Since $V \upharpoonright \mathcal{H} = (V \upharpoonright \mathcal{H})^* \in \mathfrak{B}(\mathcal{H})$, also $H_1$ is self-adjoint on $\mathcal{D}$. Now for any $z \in \mathbb{H}_\pm$, the operator $1 + VR_0(z)$ is invertible in $\mathfrak{B}(\mathcal{X})$. (Otherwise, since $VR_0(z) \in \mathrm{FA}(\mathcal{X}, \mathcal{X})$, the Fredholm alternative yields a $\psi \in \mathcal{X}\backslash\{0\}$ in the kernel of $1 + VR_0(z)$. Then $\varphi := R_0(z)\psi \in \mathcal{D}$ fulfills $(H_0 + V)\varphi = z\varphi$ with $\mathrm{Im}\, z \neq 0$, contradicting the self-adjointness of $H_1$.) By analytic Fredholm theory [Sim15, Theorem 3.14.3], the $\mathfrak{B}(\mathcal{X})$-valued function $S : z \mapsto (1 + VR_0(z))^{-1}$ is analytic in $\mathbb{H}_\pm$. Further, for $\lambda \in U$, the norm limit $VR_0(\lambda \pm i0)$ lies in $\mathrm{FA}(\mathcal{X}, \mathcal{X})$, hence $S(\lambda \pm i0) := (1 + VR_0(\lambda \pm i0))^{-1}$ exists for $\lambda$ outside a closed null set $N$, cf. [Yaf92, Sec. 1.8.3]; and (local Hölder) continuity of $VR_0(\cdot)$ on $\mathbb{H}_\pm \cup U$ translates to (local Hölder) continuity of $S$ there. Finally, the resolvent identity

$$R_1(z) = R_0(z) - R_1(z)VR_0(z) \tag{4.6}$$

implies $R_1(z) = R_0(z)S(z)$, showing that $H_1$ is $\mathcal{X}$-smooth with respect to $U\backslash N$ rather than $U$. $\qquad\square$

Now for a smooth scattering system, the wave operators are known to exist automatically, and we can express them in terms of the operators $A_j(\lambda)$:

**Proposition 4.5.** *Let $(H_0, H_1, \mathcal{X}, \mathcal{H})$ be a smooth scattering system. Then the wave operators $W_\pm(H_1, H_0)$ exist and are complete. We have for real Hölder continuous functions $\chi_0, \chi_1$ with support in compact intervals $I_0, I_1 \subset U$ and for any $\varphi_0, \varphi_1 \in \mathcal{X}$,*

$$\langle \chi_1(H_1)\varphi_1, \left(W_\pm(H_1, H_0) - \mathbf{1}\right)\chi_0(H_0)\varphi_0 \rangle = \lim_{\epsilon \downarrow 0} \int d\lambda\, d\mu \frac{\chi_1(\mu)\chi_0(\lambda)}{\lambda - \mu \mp i\epsilon} \langle \varphi_1, A_1(\mu)V A_0(\lambda)\varphi_0 \rangle, \tag{4.7}$$

*where $V = H_1 - H_0$.*

*Proof.* The existence of wave operators follows from well-known results: If $I \subset U$ is a compact interval, and $V = V_1^* V_0$, then $\mathcal{X}$-smoothness of $H_j$ implies that $V_j(R_j(\lambda + i\epsilon) - R_j(\lambda - i\epsilon))V_j^*$ is uniformly bounded in $\mathfrak{B}(\mathcal{K})$ for $\lambda \in I$ and $|\epsilon|$ sufficiently small. Hence the operators $V_j E_j(I)$ are Kato-smooth with respect to $H_j$ [Yaf92, Theorem 4.3.10]. This suffices to show that the wave operators $W_\pm(H_1, H_0)$ and $W_\pm(H_0, H_1)$ exist, since $U$ is of full measure [Yaf92, Corollary 4.5.7].

Given the existence of the wave operators, they have the "stationary" representations [Yaf92, Lemma 2.7.1]

$$\langle \psi_1, W_\pm(H_1, H_0)\psi_0 \rangle = \lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \int \langle \psi_1, R_1(\lambda \mp i\epsilon)R_0(\lambda \pm i\epsilon)\psi_0 \rangle \, d\lambda \tag{4.8}$$

for all $\psi_j \in P_j^{\mathrm{ac}}\mathcal{H}$. On the other hand, for every $\epsilon > 0$,

$$\langle \psi_1, \psi_0 \rangle = \frac{\epsilon}{\pi} \int \langle \psi_1, R_0(\lambda \mp i\epsilon)R_0(\lambda \pm i\epsilon)\psi_0 \rangle \, d\lambda. \tag{4.9}$$

Hence if $\varphi_j \in \mathcal{X}$ and $\mathrm{supp}\,\chi_j \subset I_j \subset U$, then it follows from (4.8), (4.9) and from the resolvent identity (4.6) that

$$\langle \chi_1(H_1)\varphi_1, \left(W_\pm(H_1, H_0) - \mathbf{1}\right)\chi_0(H_0)\varphi_0 \rangle$$
$$= -\lim_{\epsilon \downarrow 0} \frac{\epsilon}{\pi} \int \langle \varphi_1, \chi_1(H_1)R_1(\lambda \mp i\epsilon)VR_0(\lambda \mp i\epsilon)R_0(\lambda \pm i\epsilon)\chi_0(H_0)\varphi_0 \rangle \, d\lambda$$
$$= -\lim_{\epsilon \downarrow 0} \int \frac{\chi_1(\mu_1)}{\mu_1 - \lambda \pm i\epsilon} \frac{\chi_0(\mu_0)\epsilon/\pi}{(\mu_0 - \lambda)^2 + \epsilon^2} \langle \varphi_1, A_1(\mu_1)V A_0(\mu_0)\varphi_0 \rangle \, d\lambda\, d\mu_1\, d\mu_0 \tag{4.10}$$
$$= \lim_{\epsilon \downarrow 0} \int \langle \Phi_1(\lambda, \epsilon), \Phi_0(\lambda, \epsilon) \rangle_\mathcal{K} \, d\lambda,$$

where (4.4) has been used twice, and where $\Phi_j$ are the $\mathcal{K}$-valued functions

$$\Phi_1(\lambda, \epsilon) = \int \frac{\chi_1(\mu)}{\lambda - \mu \pm i\epsilon} V_1 A_1(\mu)\varphi_1 \, d\mu, \quad \Phi_0(\lambda, \epsilon) = \int \frac{\chi_0(\mu)\epsilon/\pi}{(\mu - \lambda)^2 + \epsilon^2} V_0 A_0(\mu)\varphi_0 \, d\mu. \tag{4.11}$$

Note that these integrals exist as they run over the compact intervals $I_j$ where the integrand is continuous in the norm of $\mathcal{K}$. As $\epsilon \to 0$, the first integral $\Phi_1(\lambda, \varepsilon)$ has a limit by the Privalov-Korn theorem (Lemma A.1), whereas the second one evidently satisfies $\Phi_0(\lambda, \epsilon) \to f_0(\lambda)V_0 A_0(\lambda)\varphi_0$.

Using the estimate from the Privalov-Korn theorem on $\Phi_1$ and an elementary estimate on $\Phi_0$, it moreover follows that $\|\Phi_1(\lambda, \epsilon)\|_\mathcal{K}\|\Phi_0(\lambda, \epsilon)\|_\mathcal{K}$ has an $\epsilon$-independent upper bound that is integrable in $\lambda$. Hence we may use dominated convergence to conclude the claimed result. $\qquad\square$

## 4.2 High energy behaviour

We start with the high-energy behaviour of a single self-adjoint operator $H$.

**Definition 4.6.** *Let $\beta \in (0,1)$. We say that an $\mathcal{X}$-smooth operator $H$ is of high-energy order $\beta$ if there exist $\hat{\lambda}, b > 0$ such that*

$$\|R(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \le b|\lambda|^{-\beta} \quad \text{for all } \lambda \in U,\ |\lambda| \ge \hat{\lambda}. \tag{4.12}$$

By (4.2), also $A(\lambda)$ then fulfills a similar estimate. Vice versa, we can deduce the high-energy behaviour of $R(\lambda \pm i0)$ from that of $A(\lambda)$, given a uniform Hölder estimate.

**Proposition 4.7.** *Let $H$ be a self-adjoint operator, $U \subset \mathbb{R}$ an open set of full measure and $A : U \to \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ be such that $\frac{d}{d\lambda}\langle\varphi, E(\lambda)\varphi\rangle = \langle\varphi, A(\lambda)\varphi\rangle$ for all $\varphi \in \mathcal{X}$, $\lambda \in U$. Suppose that $\lambda \mapsto \|A(\lambda)\|_{\mathcal{X}, \mathcal{X}^*}$ is locally integrable, that $A(\lambda)$ is locally Hölder continuous, and that there are constants $c > 0$, $\beta, \theta \in (0,1)$, $\hat{\lambda} > 0$, $q > 1$ such that $(-\infty, -\hat{\lambda}] \cup [\hat{\lambda}, \infty) \subset U$ and*

$$\|A(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \le c|\lambda|^{-\beta} \qquad\qquad\qquad \text{whenever } |\lambda| \ge \hat{\lambda}, \tag{4.13}$$

$$\|A(\lambda) - A(\lambda')\|_{\mathcal{X}, \mathcal{X}^*} \le c|\lambda|^{-\beta-\theta}|\lambda - \lambda'|^\theta \qquad \text{whenever } |\lambda| \ge \hat{\lambda},\ 1 < \lambda'/\lambda \le q^2. \tag{4.14}$$

*Then $H$ is $\mathcal{X}$-smooth and of high energy order $\beta$.*

*Proof.* $H$ is $\mathcal{X}$-smooth by Lemma 4.2. For more quantitative estimates, fix $\lambda \ge q\hat{\lambda}$ (the case $\lambda \le -q\hat{\lambda}$ is analogous). Let $I = [\lambda/q, q\lambda] \subset U$. For $\epsilon > 0$, we can write in the sense of weak integrals on $\mathcal{X} \times \mathcal{X}$,

$$R(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} \underbrace{\int_I \frac{A(\lambda')d\lambda'}{\lambda' - \lambda - i\epsilon}}_{=:J_1(\lambda)} \cdot + \underbrace{\int_{|\lambda'| \ge \hat{\lambda}, \lambda' \notin I} \frac{A(\lambda')d\lambda'}{\lambda' - \lambda}}_{=:J_2(\lambda)} + \underbrace{E(-\hat{\lambda}, \hat{\lambda})(H - \lambda)^{-1}}_{=:J_3(\lambda)} \cdot \tag{4.15}$$

To estimate these terms, we note that by our hypothesis,

$$\sup_{\lambda' \in I}\|A(\lambda')\|_{\mathcal{X}, \mathcal{X}^*} + \sup_{\lambda' \ne \lambda'' \in I} \frac{\lambda^\theta}{|\lambda' - \lambda''|^\theta}\|A(\lambda') - A(\lambda'')\|_{\mathcal{X}, \mathcal{X}^*} \le c_1\lambda^{-\beta} \tag{4.16}$$

with a constant $c_1 > 0$. Hence the Privalov-Korn theorem (Lemma A.1) yields the estimate

$$\|J_1(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \le c_1 k_\theta \lambda^{-\beta}. \tag{4.17}$$

with constants independent of $\lambda$. For estimating $J_2$, we split the integration region further into $(-\infty, -\lambda] \cup (-\lambda, -\hat{\lambda}) \cup (\hat{\lambda}, \lambda/q) \cup [q\lambda, \infty)$. We have

$$\left\| \int_{-\infty}^{-\lambda} d\lambda' \frac{A(\lambda')}{\lambda - \lambda'} \right\|_{\mathcal{X}, \mathcal{X}^*} \le \int_{-\infty}^{-\lambda} d\lambda' \frac{c|\lambda'|^{-\beta}}{|\lambda'|} \le c_1 \lambda^{-\beta} \tag{4.18}$$

with some $c_2 > 0$. A similar estimate holds for the integral over $[q\lambda, \infty)$. Further,

$$\left\| \int_{-\lambda}^{-\hat{\lambda}} d\lambda' \frac{A(\lambda')}{\lambda - \lambda'} \right\|_{\mathcal{X}, \mathcal{X}^*} \le \frac{1}{\lambda} \int_{-\lambda}^{-\hat{\lambda}} d\lambda' c|\lambda'|^{-\beta} \le c_3 \lambda^{-\beta} + c_4 \lambda^{-1} \tag{4.19}$$

with $c_3, c_4 > 0$. The interval $(\hat{\lambda}, \lambda/q)$ is handled similarly. Therefore, $\|J_2\|_{\mathcal{X}, \mathcal{X}^*} \le c_5 \lambda^{-\beta}$. Finally, it is clear that $\|J_3\|_{\mathcal{X}, \mathcal{X}^*} \le c_6\|J_3\|_{\mathcal{H}, \mathcal{H}} \le c_6(\lambda - \hat{\lambda})^{-1}$.

Combined, we have shown that $\|R(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \le c_7|\lambda|^{-\beta}$ for $|\lambda| \ge q\hat{\lambda}$, and therefore $H$ is of high-energy order $\beta$. $\qquad\square$

Now turning to smooth scattering systems, it turns out that both operators $H_j$ are always of the same high-energy order, hence we can meaningfully speak of the *system* having high-energy order $\beta$.

**Proposition 4.8.** *Let $(H_0, H_1, \mathcal{X}, \mathcal{H})$ be a smooth scattering system, and $\beta \in (0,1)$. Then $H_0$ is of high-energy order $\beta$ if and only if $H_1$ is.*

*Proof.* Let $H_0$ be of high-energy order $\beta$, and set $V := H_1 - H_0$. After possibly increasing $\hat{\lambda}$, we may assume that $\|R_0(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \leq (2\|V\|_{\mathcal{X}^*, \mathcal{X}})^{-1}$ for $|\lambda| \geq \hat{\lambda}$. Hence in this range, $\|VR_0(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}} \leq \frac{1}{2}$, and the Neumann series

$$(\mathbf{1} + VR_0(\lambda \pm i0))^{-1} = \sum_{n=0}^{\infty} \big( -VR_0(\lambda \pm i0) \big)^n \tag{4.20}$$

converges in $\mathfrak{B}(\mathcal{X}, \mathcal{X})$; that is, $\mathbf{1} + VR_0(\lambda \pm i0)$ has an inverse in $\mathfrak{B}(\mathcal{X}, \mathcal{X})$, with norm at most 2. Now from the resolvent equation (4.6), we obtain

$$R_1(\lambda \pm i0) = R_0(\lambda \pm i0)(\mathbf{1} + VR_0(\lambda \pm i0))^{-1} \tag{4.21}$$

and hence

$$\|R_1(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \leq 2\|R_0(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \leq 2b|\lambda|^{-\beta} \tag{4.22}$$

as claimed. The other direction follows by symmetric arguments. $\qquad\square$

We will now turn our attention to the high-energy behaviour of the wave operator in a smooth scattering system, and investigate whether $H_0 \sim_f H_1$. Here we will restrict to the choice $f(\lambda) = (1 + \lambda^2)^{\beta/2}$ with some $\beta \in (0,1)$, and write $H_0 \sim_\beta H_1$ as a shorthand (mutual $\beta$-boundedness).

**Theorem 4.9.** *Let $(H_0, H_1, \mathcal{X}, \mathcal{H})$ be a smooth scattering system of high-energy order $\beta \in (0,1)$. Then, $H_0 \sim_\beta H_1$.*

*Proof.* First, let $\varphi_0, \varphi_1 \in \mathcal{X}$, and let $\chi_0, \chi_1 : \mathbb{R}_+ \to [0,1]$ be continuous such that $\operatorname{supp} \chi_j \subset \cup_k I_k$, where $I_k$ are finitely many disjoint compact intervals and $I_k \subset U \cap [\hat{\lambda}, \infty)$. By Proposition 4.5, we have

$$
\begin{aligned}
\big\langle \chi_1(H_1)\varphi_1, &\big(W_\pm(H_1, H_0) - \mathbf{1}\big)H_0^\beta \chi_0(H_0)\varphi_0 \big\rangle \\
&= \lim_{\epsilon \downarrow 0} \int_0^\infty d\lambda\, d\mu \frac{\chi_1(\mu)\chi_0(\lambda)\lambda^\beta}{\lambda - \mu \mp i\epsilon} \langle \varphi_1, A_1(\mu)VA_0(\lambda)\varphi_0 \rangle \\
&= \lim_{\epsilon \downarrow 0} \int_0^\infty d\lambda\, d\mu \frac{(\lambda/\mu)^{\beta/2}}{\lambda - \mu \mp i\epsilon} \langle \Phi_1(\mu), \Phi_0(\lambda) \rangle_\mathcal{K},
\end{aligned} \tag{4.23}
$$

where $V = V_1^* V_0$ with $V_j : \mathcal{X}^* \to \mathcal{K}$, and $\Phi_j$ are the $\mathcal{K}$-valued functions on $\mathbb{R}_+$ given by

$$\Phi_j(\lambda) = \chi_j(\lambda)\lambda^{\beta/2}V_j A_j(\lambda)\varphi_j. \tag{4.24}$$

From our assumption on the high-energy order of the $H_j$, we have $\|A_j(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \leq \frac{b}{\pi}\lambda^{-\beta}$ for all $\lambda \in U$ with $|\lambda| \geq \hat{\lambda}$. Hence, using (4.5),

$$
\begin{aligned}
\int_0^\infty d\lambda \|\Phi_j(\lambda)\|_\mathcal{K}^2 &\leq \int_{\hat{\lambda}}^\infty d\lambda\, \chi_j(\lambda)^2 \lambda^\beta \|V_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \|A_j(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \langle \varphi_j, A_j(\lambda)\varphi_j \rangle \\
&\leq \frac{b}{\pi}\|V_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \int_{\hat{\lambda}}^\infty d\lambda\, \chi_j(\lambda)^2 \langle \varphi_j, A_j(\lambda)\varphi_j \rangle \leq \frac{b}{\pi}\|V_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \|\varphi_j\|_\mathcal{H}^2.
\end{aligned} \tag{4.25}
$$

In other words, the $\Phi_j$ are elements of $L^2(\mathbb{R}_+, \mathcal{K})$, and the constant $b > 0$ in their norm is independent of our choice of $\chi_j$ under the given constraints. Now in (4.23), $K(\lambda, \mu) = \frac{(\lambda/\mu)^{\beta/2}}{\lambda - \mu \mp i0}$ is the kernel of a bounded operator $T$ on $L^2(\mathbb{R}_+)$ by Lemma C.1. Hence also $T \otimes \mathbf{1}_\mathcal{K}$ is bounded on $L^2(\mathbb{R}_+, \mathcal{K})$. This yields

$$
\begin{aligned}
|\langle \varphi_1, \chi_1(H_1)(W_\pm(H_1, H_0) - \mathbf{1})H_0^\beta \chi_0(H_0)\varphi_0 \rangle| &\leq c\|\Phi_0\|_{L^2(\mathbb{R}_+, \mathcal{K})}\|\Phi_1\|_{L^2(\mathbb{R}_+, \mathcal{K})} \\
&\leq \frac{cb}{\pi}\|V_0\|_{\mathcal{X}^*, \mathcal{K}}\|V_1\|_{\mathcal{X}^*, \mathcal{K}}\|\varphi_0\|_\mathcal{H}\|\varphi_1\|_\mathcal{H}
\end{aligned} \tag{4.26}
$$

with a universal $c > 0$ (depending only on $\beta$).

Now we can choose a sequence of $\chi_j$ of the form stated above such that $\chi_j(H_j)$ converges strongly to $P_j^{\text{ac}} E_j(\hat{\lambda}, \infty)$. Thus (4.26) yields, considering that $E_j(\hat{\lambda}, \infty)(H_0^\beta - (1 + H_0^2)^{\beta/2})$ is bounded,

$$E_1(\hat{\lambda}, \infty) P_1^{\text{ac}} \big( W_\pm(H_1, H_0) - \mathbf{1} \big)(1 + H_0^2)^{\beta/2} P_0^{\text{ac}} E_0(\hat{\lambda}, \infty) \in \mathfrak{B}(\mathcal{H}). \tag{4.27}$$

Similar arguments show that the analogous expressions with one or both of the $E_j(\hat{\lambda}, \infty)$ swapped for $E_j(-\infty, -\hat{\lambda})$ are bounded. (This requires boundedness of the integral operator with kernel $K(\lambda, \mu) = \frac{(\lambda/\mu)^{\beta/2}}{\mu + \lambda}$, see again Lemma C.1.) Moreover, as Lemma B.1 shows, the boundedness of $H_0 - H_1$ implies that also $(1 + H_0^2)^{\beta/2} - (1 + H_1^2)^{\beta/2}$ is bounded. Hence Lemma 2.2 is applicable, and we obtain that $P_1^{\text{ac}}(W_\pm(H_1, H_0) - \mathbf{1})(1 + H_0^2)^{\beta/2} P_0^{\text{ac}}$ is bounded. The statement with $H_1$ and $H_0$ exchanged follows symmetrically. $\qquad\square$

## 4.3 Pseudo-differential operators

Our results in the smooth method can be applied to a wide range of examples where $H_0$ is a differential or pseudo-differential operator and the perturbation $V = H_1 - H_0$ is a multiplication operator. Here we treat mutual $\beta$-boundedness for the perturbed polyharmonic operator, i.e., where $H_0$ is a fractional power of the Laplace operator, using familiar techniques for the Schrödinger operator ($\ell = 2$ below); see, e.g., [Yaf10].

**Example 4.10.** *Let $H_0 = (-\Delta)^{\ell/2}$ acting on its natural domain of self-adjointness in $\mathcal{H} = L^2(\mathbb{R}^n)$, where $\ell \in (1, \infty)$, $n \in \mathbb{N}$. Let $v \in L^\infty(\mathbb{R}^n)$ such that $\sup_x (1 + |x|^2)^\alpha |v(x)| < \infty$ with some $\alpha > \frac{1}{2}$, and let $V \in \mathfrak{B}(\mathcal{H})$ be the multiplication with $v$. Then $H_0$ and $H_1 := H_0 + V$ are mutually $\beta$-bounded for any $0 < \beta \le 1 - \frac{1}{\ell}$.*

*Proof.* Let $\langle x \rangle$ be the multiplication operator by $(1 + |x|^2)^{1/2}$. We define $\mathcal{X} \subset \mathcal{H}$ as the completion of $\mathscr{S}(\mathbb{R}^n)$ in the norm $\|f\|_{\mathcal{X}} := \|\langle x \rangle^\alpha f\|_{\mathcal{H}}$. To show that $H_0$ is $\mathcal{X}$-smooth, let us introduce for fixed $\lambda > 0$ the map $\Gamma(\lambda) : \mathscr{S}(\mathbb{R}^n) \to L^2(S^{n-1})$ given by

$$\big( \Gamma(\lambda) f \big)(\omega) = (2\pi)^{-n/2} \int d^n x \, e^{i\langle \lambda\omega, x \rangle} f(x) = \tilde{f}(\lambda\omega). \tag{4.28}$$

By [Yaf10, Theorem 1.1.4] it extends to a bounded operator $\Gamma(\lambda) : \mathcal{X} \to L^2(S^{n-1})$ with norm bound

$$\|\Gamma(\lambda)\|_{\mathcal{X}, L^2(S^{n-1})} \le C \lambda^{-\frac{n-1}{2}} \tag{4.29}$$

for all $\lambda > 0$, where $C$ is independent of $\lambda$. It also follows from [Yaf10, Theorem 1.1.5] that $\Gamma(\lambda)$ is locally Hölder continuous, in the sense that

$$\|\Gamma(\lambda) - \Gamma(\lambda')\|_{\mathcal{X}, L^2(S^{n-1})} \le C |\lambda - \lambda'|^\theta \tag{4.30}$$

for some $\theta \in (0, 1)$, where $C$ can be chosen uniformly for all $\lambda, \lambda'$ in a fixed compact interval in the open half line $\mathbb{R}_+$. The derivative of the spectral measure of $H_0$ is now given by

$$\frac{d}{d\lambda} \langle f, E_0(\lambda) g \rangle = \frac{1}{\ell} \lambda^{\frac{n}{\ell} - 1} \langle f, \Gamma(\lambda^{1/\ell})^* \Gamma(\lambda^{1/\ell}) g \rangle =: \langle f, A_0(\lambda) g \rangle \tag{4.31}$$

for $f, g \in \mathcal{X}$ and $\lambda > 0$, and $A_0(\lambda) = 0$ for $\lambda < 0$. As a consequence of (4.29), $A_0(\lambda)$ is a bounded operator from $\mathcal{X}$ to $\mathcal{X}^*$ with norm bounded by

$$\|A_0(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \le \hat{C} \lambda^{-1 + \frac{1}{\ell}}, \quad \lambda > 0, \tag{4.32}$$

and (4.30) implies that $A_0(\lambda)$ is locally Hölder continuous on $\mathbb{R}_+$ as a composition of Hölder continuous functions. Therefore all the requirements of Lemma 4.2 are satisfied with $U = \mathbb{R} \backslash \{0\}$, and we can conclude that $H_0$ is $\mathcal{X}$-smooth.

By Lemma 4.4, also $H_1$ is then $\mathcal{X}$-smooth if we can show that $R_0(z) \in \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ is compact for $\operatorname{Im} z \neq 0$ (note that $\mathcal{X}$ is hilbertisable). But this is equivalent to compactness of $\langle x \rangle^{-\alpha} R_0(z) \langle x \rangle^{-\alpha}$ in $\mathfrak{B}(\mathcal{H})$, which follows since this operator is a product of suitable multiplication and convolution operators [Yaf92, Lemma 1.6.5].

For analyzing the high-energy behaviour of $H_0$, we define the unitary dilation operators on $\mathcal{H}$,

$$\big(D(\tau)f\big)(x) = \tau^{-n/2} f(\tau^{-1}x), \quad f \in \mathcal{H}, \ \tau > 0. \tag{4.33}$$

Considering them as operators from $\mathcal{X}$ to $\mathcal{X}$, or from $\mathcal{X}^*$ to $\mathcal{X}^*$, one finds that

$$\|D(\tau)\|_{\mathcal{X},\mathcal{X}} \leq C\tau^\alpha \quad \text{and} \quad \|D(\tau^{-1})\|_{\mathcal{X}^*,\mathcal{X}^*} \leq C\tau^\alpha \quad \text{for all } \tau \geq 1. \tag{4.34}$$

One also computes that

$$D(\tau^{-1}) R_0(z) D(\tau) = \tau^\ell R_0(\tau^\ell z). \tag{4.35}$$

Together with (4.34), we then find

$$\|R_0(\lambda \pm i0)\|_{\mathcal{X},\mathcal{X}^*} \leq C^2 \|R_0(1 \pm i0)\|_{\mathcal{X},\mathcal{X}^*} \, \lambda^{-1+\frac{2\alpha}{\ell}} \tag{4.36}$$

and conclude that $H_0$ is of high-energy order $\beta = 1 - \frac{2\alpha}{\ell}$, provided this is positive. By Theorem 4.9, we then have $H_0 \sim_\beta H_1$. Since $\alpha > \frac{1}{2}$ was arbitrary and the $\mathcal{X}$-norm becomes stronger with increasing $\alpha$, we have thus shown our claim for all $0 < \beta < 1 - \frac{1}{\ell}$.

(Alternatively, we may deduce this as follows: Pick some interval $[1, q^2]$ where $A_0(\lambda)$ is uniformly Hölder continuous. Using the formula $D(\tau^{-1}) A_0(\lambda) D(\tau) = \tau^\ell A_0(\tau^\ell \lambda)$ and (4.34), a scaling argument like above shows that the hypothesis of Proposition 4.7 is satisfied, yielding the result.)

Proving the claim for $\beta = 1 - \frac{1}{\ell}$ requires more effort; we sketch the argument. We make use of the Agmon-Hörmander space $\mathcal{B} \subset \mathcal{H}$; see [Yaf10, Secs. 6.3 and 7.1] for its definition and properties. Here we need only that $\mathcal{X} \subset \mathcal{B} \subset \mathcal{H}$ are continuous dense inclusions, and that, in some improvement over (4.34),

$$\|D(\tau)\|_{\mathcal{B},\mathcal{B}} \leq C\tau^{1/2} \quad \text{and} \quad \|D(\tau^{-1})\|_{\mathcal{B}^*,\mathcal{B}^*} \leq C\tau^{1/2} \quad \text{for all } \tau \geq 1. \tag{4.37}$$

We will show below that $R_0(\lambda \pm i0)$ is bounded from $\mathcal{B}$ to $\mathcal{B}^*$ for each fixed $\lambda > 0$. A scaling argument as above then yields

$$\|R_0(\lambda \pm i0)\|_{\mathcal{B},\mathcal{B}^*} \leq C^2 \|R_0(1 \pm i0)\|_{\mathcal{B},\mathcal{B}^*} \, \lambda^{-1+\frac{1}{\ell}} \tag{4.38}$$

for all $\lambda \geq 1$. An analogous estimate holds for $\|R_0(\lambda \pm i0)\|_{\mathcal{X},\mathcal{X}^*}$, since the inclusion $\mathcal{X} \subset \mathcal{B}$ is continuous. Hence $H_0$ is of high-energy order $\beta = 1 - \frac{1}{\ell}$, and $H_0 \sim_\beta H_1$.

In order to show that $R_0(\lambda \pm i0) \in \mathfrak{B}(\mathcal{B}, \mathcal{B}^*)$, let us define for $\epsilon > 0$,

$$S_{\pm\epsilon} := L R_0^{(2)}(\lambda^{2/\ell} \pm i\epsilon) \in \mathfrak{B}(\mathcal{H}), \tag{4.39}$$

where $R_0^{(2)}(\cdot)$ is the resolvent of $-\Delta$, and $L$ is the multiplication operator in Fourier space by the function

$$\hat{\ell}(\xi) := \frac{|\xi|^2 - \lambda^{2/\ell}}{|\xi|^\ell - \lambda}. \tag{4.40}$$

One notices that $\langle f, S_{\pm\epsilon} g \rangle \to \langle f, R_0(\lambda \pm i0) g \rangle$ for each $f, g \in \mathscr{S}(\mathbb{R}^n) \subset \mathcal{B}$ as $\epsilon \to 0$. On the other hand, [Yaf10, Theorem 6.3.3] yields that $\|S_{\pm\varepsilon}\|_{\mathcal{B},\mathcal{B}^*}$ is uniformly bounded for all $\epsilon \leq 1$.[4] Hence, $|\langle f, R_0(\lambda \pm i0) g \rangle| \leq c \|f\|_{\mathcal{B}} \|g\|_{\mathcal{B}}$ for all $f, g \in \mathscr{S}(\mathbb{R}^n)$, and $R_0(\lambda \pm i0)$ extends to a bounded operator from $\mathcal{B}$ to $\mathcal{B}^*$. $\qquad \square$

The estimates on the high-energy order, $0 < \beta \leq 1 - \frac{1}{\ell}$, cannot be improved in general, as the following special case shows.

**Example 4.11.** *In Example 4.10, let $n = 1$, $\ell = 2$, and suppose that $v \neq 0$ is compactly supported and nonnegative. Then $H_0$ and $H_1 = H_0 + V$ are* not *mutually $\beta$-bounded for any $\beta > \frac{1}{2}$.*

---

[4]Theorem 6.3.3 in [Yaf10] assumes that the function $\hat{\ell}$ is smooth everywhere, but this is not essential for its proof; it suffices that, as in our case, the function is smooth outside the origin $\xi = 0$, and bounded in a neighbourhood of the origin.

*Proof.* First, note that $P_0^{\mathrm{ac}} = \mathbf{1}$; also, since $v \geq 0$, we know that $H_1 \geq 0$ and hence $H_1$ does not have eigenvalues [Yaf10, Lemma 6.2.1], i.e., $P_1^{\mathrm{ac}} = \mathbf{1}$.

Now let $v$ be supported in the compact interval $[a, b]$. We choose $\varphi \in C_0^\infty(b, \infty)$ and $\psi \in L^2(\mathbb{R})$ with its Fourier transform $\tilde{\psi} \in C_0^\infty(\mathbb{R}_+)$ such that $\langle \varphi, \psi \rangle \neq 0$. For $n \in \mathbb{N}$, define $\varphi_n(x) := e^{inx}\varphi(x)$, $\psi_n(x) := e^{inx}\psi(x)$. The wave operator $W := W_\pm(H_1, H_0)$ can in our case be written as

$$(W\psi)(x) = \frac{1}{\sqrt{2\pi}} \int dk \, m(x, k) T(k) e^{ikx} \tilde{\psi}(k) \tag{4.41}$$

with a complex-valued function $T$ and a certain integral kernel $m$, which for $x > b$ is given by $m(x, k) = 1$ [DT79, Sec. 2]. Hence we have

$$\langle \varphi_n, (W - \mathbf{1})(1 + H_0^2)^{\beta/2}\psi_n \rangle = \int dx \, \overline{\varphi_n(x)} \int dk \, \big(m(x, k)T(k) - 1\big) e^{ikx}(1 + k^4)^{\beta/2}\tilde{\psi}_n(k)$$

$$= \int dk \, \overline{\tilde{\varphi}(k)} \big(T(k + n) - 1\big)(1 + (k + n)^4)^{\beta/2}\tilde{\psi}(k). \tag{4.42}$$

Now by [DT79, Proof of Thm. 1.IV], $T$ has the asymptotics

$$T(k) = 1 + \frac{\int dx \, v(x)}{2ik} + O(k^{-2}) \quad \text{as } k \to \infty, \tag{4.43}$$

where $\int dx \, v(x) \neq 0$ by hypothesis. Since $\beta > 1/2$, we find that (4.42) diverges as $n \to \infty$, while $\|\varphi_n\|$ and $\|\psi_n\|$ are independent of $n$. $\qquad\square$

## 4.4 Tensor products

We now ask whether the high-energy order of an operator is stable under taking tensor products, in the following sense: Let $H_A$ be a self-adjoint operator on $\mathcal{H}_A$ which is smooth with respect to some Gelfand triple $\mathcal{X}_A \subset \mathcal{H}_A \subset \mathcal{X}_A^*$. Let $H_B$ be another self-adjoint operator on a Hilbert space $\mathcal{H}_B$, assumed to have purely discrete spectrum. On $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$, consider $H := H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$. (In applications in Physics, which we will discuss further below, $H_A$ is typically some differential operator on an $L^2$ space, and $\mathcal{H}_B$ describes some "inner degrees of freedom".)

In the following, we will always denote the resolvent of $H_A$ as $R_A(z)$, etc. We note that, if $H_B = \sum_j \lambda_j P_j$ is the spectral decomposition of $H_B$, then

$$R(z) = \sum_j R_A(z - \lambda_j) \otimes P_j. \tag{4.44}$$

at least weakly on $\mathcal{H} \times \mathcal{H}$; cf. [BA11, Sec. 5.1]. The same relation then holds with $z$ replaced with $\lambda \pm i0$ as long as $\lambda - \lambda_j \in U_A$ for all $j$, and at least in the sense of matrix elements between vectors of the form $\psi_A \otimes \psi_B$ where $\psi_A \in \mathcal{X}_A$ and $\psi_B$ is an eigenvector of $H_B$. (We will clarify below when the limit exists in the norm sense.)

For simplicity, we first treat the case of a finite-dimensional space $\mathcal{H}_B$.

**Proposition 4.12.** *Let $H_A$ be a self-adjoint operator on a Hilbert space $\mathcal{H}_A$ which is $\mathcal{X}_A$-smooth with respect to a set $U_A$ such that $\mathbb{R} \backslash U_A$ is finite, and let $H_A$ be of high-energy order $\beta \in (0, 1)$. Let $H_B$ be a self-adjoint operator on a finite dimensional Hilbert space $\mathcal{H}_B$.*

*Set $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{X} := \mathcal{X}_A \otimes \mathcal{H}_B \subset \mathcal{H}$. Then $H = H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$ is $\mathcal{X}$-smooth and of high-energy order $\beta$.*

*Proof.* Set $N := \mathbb{R} \backslash U_A$ and $S := \sigma(H_B) + N$ (both finite sets), and let $U := \mathbb{R} \backslash S$. We will show that $R$ is locally Hölder continuous on $U \pm i[0, \infty)$; clearly it suffices to show this in a neighbourhood of each real point.

Since $U$ is open and its complement finite, we can for any given $\lambda \in U$ find a neighbourhood $V_\lambda = [\lambda - \delta, \lambda + \delta] \pm i[0, \epsilon]$ ($\delta, \epsilon > 0$) such that $V_\lambda - \lambda_j \subset U_A \pm i[0, \infty)$ for all $j$. We now employ (4.44) and use local Hölder continuity of $R_A$ to estimate

$$\|R(z') - R(z'')\|_{\mathcal{X}, \mathcal{X}^*} \leq \sum_j \|R_A(z' - \lambda_j) - R_A(z'' - \lambda_j)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq \sum_j c_j |z' - z''|^{\theta_j} \tag{4.45}$$

for all $z', z'' \in V_\lambda$, with constants $c_j > 0$, $\theta_j \in (0,1)$. Since the sum is finite, this proves local Hölder continuity of $R$. In particular, the limits $R(\lambda \pm i0)$ exist in $\mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ for all $\lambda \in U$. Hence $H$ is $\mathcal{X}$-smooth.

For the high-energy behaviour of $R$, we similarly estimate for sufficiently large $|\lambda|$ using the high-energy order of $R_A$,

$$\|R(\lambda \pm i0)\|_{\mathcal{X}, \mathcal{X}^*} \leq \sum_j \|R_A(\lambda - \lambda_j \pm i0)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq \sum_j c_j |\lambda - \lambda_j|^{-\beta} \leq c' |\lambda|^{-\beta} \tag{4.46}$$

which shows that $H$ is also of high-energy order $\beta$. $\qquad\square$

We now allow for an infinite-dimensional space $\mathcal{H}_B$. This requires stronger uniformity assumptions on our bounds on $R_A$, as well as some restrictions on the spectrum of $H_B$. In order to avoid technical complications with the tensor product, we also assume that $\mathcal{X}_A$ is a Hilbert space.

**Theorem 4.13.** *Let $\mathcal{X}_A \subset \mathcal{H}_A \subset \mathcal{X}_A^*$ a Gelfand triple with a Hilbert space $\mathcal{X}_A$, and $H_A$ a self-adjoint operator on $\mathcal{H}_A$. Let $H_B$ be another self-adjoint operator on a Hilbert space $\mathcal{H}_B$. Suppose that:*

*(a) $H_A$ is $\mathcal{X}_A$-smooth, with respect to a set $U_A$ which has finite complement.*

*(b) There exist $\Lambda > 0$, $\theta \in (0,1)$, $c > 0$ and $\epsilon > 0$ such that*

$$\|R_A(z) - R_A(z')\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq c|z - z'|^\theta \tag{4.47}$$

*whenever $|\operatorname{Re} z^{(\prime)}| \geq \Lambda$, $\pm \operatorname{Im} z^{(\prime)} \in [0, \epsilon]$, and $|z - z'| \leq 1$.*

*(c) There exist $c > 0$ and $\beta \in (0,1)$ such that for all $\lambda \in U_A$,*

$$\|R_A(\lambda \pm i0)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq c(1 + \lambda^2)^{-\beta/2}. \tag{4.48}$$

*(d) There exists $\gamma > 0$ such that $(1 + H_B^2)^{-\gamma + \beta/2}$ is of trace class.*

*Set $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ and let $\mathcal{X} \subset \mathcal{H}$ be the Hilbert space with the following norm:*

$$\| \cdot \|_{\mathcal{X}} = \| \cdot \|_{\mathcal{X}_A} \otimes \|(1 + H_B^2)^{\gamma/2} \cdot \|_{\mathcal{H}_B}. \tag{4.49}$$

*Then $H := H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$ is $\mathcal{X}$-smooth and of high-energy order $\beta$.*

*Proof.* As in the proof of Proposition 4.12, for given $\lambda$ we choose a compact complex neighbourhood $V_\lambda$ such that $V_\lambda - \lambda_j \subset U_A \pm i[0, \infty)$ for all $j$. (This is still possible since, due to (d), the $\lambda_j$ have no accumulation point.) Analogous to (4.45), but now with the modified norm (4.49), we obtain the estimate for $z', z'' \in V_\lambda$,

$$\|R(z') - R(z'')\|_{\mathcal{X}, \mathcal{X}^*} \leq \sum_j (1 + \lambda_j^2)^{-\gamma} \|R_A(z' - \lambda_j) - R_A(z'' - \lambda_j)\|_{\mathcal{X}_A, \mathcal{X}_A^*}. \tag{4.50}$$

We split the sum into those $j$ where $|\lambda - \lambda_j| \leq \Lambda + 1$ (with $\Lambda$ as in condition (b)) and their complement. Since $|\lambda_j| \to \infty$, the first mentioned sum is finite and can be estimated as in (4.46). For the remaining sum, we use (b) to show (if $V_\lambda$ was chosen sufficiently small so that $|z' - z''| \leq 1$),

$$\sum_{j:\, |\lambda - \lambda_j| \geq \Lambda} (1 + \lambda_j^2)^{-\gamma} \|R_A(z' - \lambda_j) - R_A(z'' - \lambda_j)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq \sum_j (1 + \lambda_j^2)^{-\gamma} c|z' - z''|^\theta \leq c'|z' - z''|^\theta \tag{4.51}$$

with a finite $c' > 0$, since $(1 + H_B^2)^{-\gamma}$ is trace class. In conclusion, $R(z)$ is locally Hölder continuous (also at the boundary $U \pm i0$), hence $H$ is $\mathcal{X}$-smooth.

For the high-energy order, we estimate for $\lambda \in U$ using condition (c),

$$\|R(\lambda \pm i0)\|_{\mathcal{X},\mathcal{X}^*} \leq \sum_j (1+\lambda_j^2)^{-\gamma}\|R_A(\lambda - \lambda_j \pm i0))\|_{\mathcal{X}_A,\mathcal{X}_A^*} \leq c\sum_j (1+(\lambda-\lambda_j)^2)^{-\beta/2}(1+\lambda_j^2)^{-\gamma}$$

$$\leq c'(1+\lambda^2)^{-\beta/2}\sum_j (1+\lambda_j^2)^{-\gamma+\beta/2}.$$

(4.52)

(We have used the inequality $\frac{1}{(x-y)^2} \leq 2\frac{1+x^2}{1+y^2}$ for $x,y \in \mathbb{R}$.) The series here is convergent due to (d). Thus $H$ is of high-energy order $\beta$. $\qquad\square$

As usual, the detailed estimates in the previous theorem can be explicitly verified only in very simple examples. However, a perturbation argument as in Lemma 4.4 allows us to extend them:

**Corollary 4.14.** *In the situation of Proposition 4.12 or Theorem 4.13, suppose that $R_A(z) \in \mathrm{FA}(\mathcal{X},\mathcal{X}^*)$ for every $z \in \mathbb{C}\backslash\mathbb{R}$. If $V = V^* \in \mathfrak{B}(\mathcal{X}^*,\mathcal{X})$, then $(H, H+V, \mathcal{X}, \mathcal{H})$ is a smooth scattering system of high-energy order $\beta$. In particular, $H \sim_\beta H+V$.*

*Proof.* Note that the spectral projectors $P_j$ of $H_B$ have finite rank; in the case of Theorem 4.13, this follows from condition (d). Thus every term $R_A(z) \otimes P_j$ in the series (4.44) lies in $\mathrm{FA}(\mathcal{X},\mathcal{X}^*)$. On the other hand, the series converges absolutely in $\mathfrak{B}(\mathcal{X},\mathcal{X}^*)$: In the first situation, this is trivial; in the second situation, note that $\|R_A(z)\|_{\mathcal{X}_A,\mathcal{X}_A^*} \leq \|R_A(z)\|_{\mathcal{H}_A,\mathcal{H}_A} \leq |\mathrm{Im}\, z|^{-1}$, and hence

$$\sum_j \|R_A(z-\lambda_j) \otimes P_j\|_{\mathcal{X},\mathcal{X}^*} \leq |\mathrm{Im}\, z|^{-1}\sum_j (1+\lambda_j^2)^{-\gamma} < \infty.$$

(4.53)

Thus $R(z) \in \mathrm{FA}(\mathcal{X},\mathcal{X}^*)$ since this space is norm-closed. The statement now follows from Lemma 4.4 and Theorem 4.9. $\qquad\square$

We will now give some more concrete examples in which our results on tensor products are applicable. In these, $H_A$ will be a differential operator and $H_B$ an operator with discrete spectrum.

**Example 4.15.** *Let $H_A = -\Delta$ acting on its natural domain of self-adjointness in $\mathcal{H}_A = L^2(\mathbb{R}^3)$ and $H_B = -\Delta$ acting on $\mathcal{H}_B = L^2(S^2)$, where $S^2$ is the two-dimensional sphere. Let $v : \mathbb{R} \to \mathfrak{B}(\mathcal{H}_B)$, $v(x) = v(x)^*$, such that*

$$\sup_x (1+|x|^2)^\alpha \left\|(1+H_B^2)^{-\gamma/2}v(x)(1+H_B^2)^{-\gamma/2}\right\|_{\mathcal{H}_B,\mathcal{H}_B} < \infty$$

(4.54)

*with some $\alpha > 1$, $\gamma > (\beta+1)/2$ and $\beta = 1/2$, and let $V \in \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be the operator multiplying with $v$. Then, with $H := H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$, we have $H \sim_\beta H+V$.*

*Proof.* We aim to apply Theorem 4.13 and Corollary 4.14; let $\mathcal{X}$ be as defined there, with $\|f\|_{\mathcal{X}_A} = \|\langle x \rangle^\alpha f\|_{\mathcal{H}_A}$. By Example 4.10, we have $\mathcal{X}_A$-smoothness of $H_A$ with $U_A = \mathbb{R}\backslash\{0\}$, so that condition (a) in the Theorem is fulfilled.

To show condition (b), it suffices to consider $\alpha < \frac{3}{2}$ and $\mathrm{Im}\, z^{(\prime)} \geq 0$. We use the dilation operators $D(\tau)$ defined in (4.33) and the relation (4.35) to show that for every $\tau \geq 1$,

$$\|R_A(z)-R_A(z')\|_{\mathcal{X}_A,\mathcal{X}_A^*} = \tau^{-2}\|D(\tau^{-1})\|_{\mathcal{X}_A^*,\mathcal{X}_A^*}\|R_A(\tau^{-2}z)-R_A(\tau^{-2}z')\|_{\mathcal{X}_A,\mathcal{X}_A^*}\|D(\tau)\|_{\mathcal{X}_A,\mathcal{X}_A}$$

$$\leq C\tau^{2\alpha-2}\|R_A(\tau^{-2}z)-R_A(\tau^{-2}z')\|_{\mathcal{X}_A,\mathcal{X}_A^*}.$$

(4.55)

Since however $R_A(z)$ is locally Hölder continuous with exponent $\theta = \alpha - \frac{1}{2}$, it fulfills a uniform Hölder estimate on, say, the compact region $[1,2] \pm i[0,1]$. Choosing $\tau = (\mathrm{Re}\, z)^{1/2}$, we thus have

$$\|R_A(z)-R_A(z')\|_{\mathcal{X},\mathcal{X}^*} \leq C(\mathrm{Re}\, z)^{\alpha-1-\theta}|z-z'|^\theta \leq C|z-z'|^\theta$$

(4.56)

whenever $1 \leq \mathrm{Re}\, z \leq \mathrm{Re}\, z' \leq 2\,\mathrm{Re}\, z$ and $0 \leq \mathrm{Im}\, z^{(\prime)} \leq 1$; likewise for $z$ and $z'$ exchanged. This includes the region $\mathrm{Re}\, z^{(\prime)} \geq 1$, $0 \leq \mathrm{Im}\, z^{(\prime)} \leq 1$, $|z-z'| \leq 1$, hence (b) follows.

For condition (c), note that $\|R_A(\lambda \pm i0)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq C|\lambda|^{-1/2}$ for $|\lambda| \geq 1$ by Example 4.10. For $|\lambda| \leq 1$ we use the fact that $\|R_A(\lambda \pm i0)\|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq C$ [Yaf10, Proposition 7.1.16]; it enters here that $\alpha > 1$ and that we consider the Laplacian on $\mathbb{R}^3$.

Regarding condition (d): Since $\sigma(H_B) = \{\ell(\ell+1)\}_{\ell \in \mathbb{N}_0}$ with degeneracy $2\ell + 1$, we can compute

$$\mathrm{tr}(1 + H_B^2)^{-\gamma+\beta/2} = \sum_{\ell \in \mathbb{N}_0} (2\ell+1)(1 + \ell^2(\ell+1)^2)^{-\gamma+\beta/2} \leq 2\sum_{\ell \in \mathbb{N}_0}(1+\ell)^{-4\gamma+2\beta+1}, \qquad (4.57)$$

which converges for $4\gamma > 2\beta + 2$ as in the hypothesis.— In conclusion, Theorem 4.13 applies. Also, we already noted in Example 4.10 that $R_A(z)$ is compact in $\mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ for $z \in \mathbb{C}\backslash\mathbb{R}$, and we have $V \in \mathfrak{B}(\mathcal{X}^*, \mathcal{X})$ by assumption (4.54). Hence we can apply Corollary 4.14 and conclude that $H \sim_\beta H + V$. $\qquad \square$

Similar methods would apply to Laplace operators in higher dimensions ($n \geq 3$), but not for $n < 3$, since in that case there is no uniform bound on $\|R_A(z)\|_{\mathcal{X}_A, \mathcal{X}_A^*}$ near $z = 0$.

Let us focus on the one-dimensional Laplacian here. Instead of the "free" operator $-\Delta$, one can consider $H_A = -\Delta + V_A$ where $V_A$ is multiplication with a nonnegative, sufficiently rapidly decaying function; its resolvent behaves better near $z = 0$, so that we can obtain a result similar to the 3-dimensional case.

**Example 4.16.** *Let $H_A = -\Delta + V_A$ acting on $\mathcal{H}_A = L^2(\mathbb{R})$, where $V_A$ is multiplication with a nonnegative, compactly supported function $v_A \in L^\infty(\mathbb{R})\backslash\{0\}$. Let $\beta \in (0, \frac{1}{2}]$, and let $H_B$ be another self-adjoint operator on a Hilbert space $\mathcal{H}_B$ such that $(1+H_B^2)^{-\gamma+\beta/2}$ is of trace class for some $\gamma > 0$. Let $v : \mathbb{R} \to \mathfrak{B}(\mathcal{H}_B)$, $v(x) = v(x)^*$, such that*

$$\sup_x (1+x^2)^\alpha \left\|(1+H_B^2)^{-\gamma/2} v(x)(1+H_B^2)^{-\gamma/2}\right\|_{\mathcal{H}_B, \mathcal{H}_B} < \infty \qquad (4.58)$$

*with some $\alpha > \frac{3}{2}$; and let $V \in \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be the operator multiplying with $v$. Then $H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B \sim_\beta H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B + V$.*

*Proof.* First, as the hypothesis becomes only stronger with increasing $\alpha$, we can assume without loss of generality that $\frac{3}{2} < \alpha < 2$.

Now let $\mathcal{X}_A \subset \mathcal{H}_A$ be once more defined by its norm $\|\cdot\|_{\mathcal{X}_A} = \|\langle x \rangle^\alpha \cdot \|_{L^2(\mathbb{R})}$, and let $\mathcal{H}$ and $\mathcal{X} \subset \mathcal{H}$ be as in Eq. (4.49). As before, we aim to verify conditions (a)–(d) of Theorem 4.13 in our situation.

For (a), we know from Example 4.10 that $H_A = -\Delta + V_A$ is $\mathcal{X}_A$-smooth. In fact, since $H_A \geq 0$ cannot have negative eigenvalues, we can choose $U_A = \mathbb{R}\backslash\{0\}$ [Yaf10, Lemma 6.2.1].

Next, we show (b) as follows. Let $R_0(z)$ be the resolvent of the negative Laplacian on $\mathcal{H}_A$. Like in (4.21), we write $R_A(z) = R_0(z)F(z)$ with $F(z) := (\mathbf{1} - G(z))^{-1}$ and $G(z) := V_A R_0(z)$, for any $z$ where the inverse exists (we will clarify this below). As in Example 4.15, $R_0$ fulfills a uniform Hölder estimate

$$\|R_0(z) - R_0(z')\|_{\mathcal{X}, \mathcal{X}^*} \leq C|z|^{\alpha-1-\theta}|z-z'|^\theta \quad \text{whenever } 1 \leq \mathrm{Re}\, z, \mathrm{Re}\, z', |z-z'| \leq 1. \qquad (4.59)$$

Since $\alpha > \frac{3}{2}$, we can choose any $\theta < 1$ here (cf. [Yaf10, Proposition 1.7.1]). An analogous Hölder estimate then holds for $G(z)$ in $\|\cdot\|_{\mathcal{X}, \mathcal{X}}$.

Further, since $\|R_0(z)\|_{\mathcal{X}, \mathcal{X}^*}$ decays at large $|z|$, we can choose $\Lambda_0 > 0$ such that $\|G(z)\|_{\mathcal{X}, \mathcal{X}} \leq \frac{1}{4}$ for all $\mathrm{Re}\, z \geq \Lambda_0$; the inverse $F(z) = (1 - G(z))^{-1}$ then exists as a convergent Neumann series, and $\|F(z)\|_{\mathcal{X}, \mathcal{X}} \leq \frac{4}{3}$. To obtain Hölder estimates for $F$, we note the identity

$$F(z) - F(z') = F(z')\Big([\mathbf{1} - (G(z) - G(z'))F(z')]^{-1} - \mathbf{1}\Big)$$
$$= F(z')\sum_{k=1}^\infty \Big((G(z) - G(z'))F(z')\Big)^k \qquad (4.60)$$

where the series converges by the above estimates. By taking norms we obtain:

$$\|F(z) - F(z')\|_{\mathcal{X}, \mathcal{X}} \leq C''\|G(z) - G(z')\|_{\mathcal{X}, \mathcal{X}} \leq C'''|z|^{\alpha-2}|z-z'|$$
$$\text{whenever } \Lambda_0 \leq \mathrm{Re}\, z, \mathrm{Re}\, z', |z-z'| \leq 1. \qquad (4.61)$$

where the factor $|z|^{\alpha-1-\theta}$ is decreasing (for suitable $\theta$, noting $\alpha < 2$). Hence, as $R_0$ is bounded in $\|\cdot\|_{\mathcal{X},\mathcal{X}^*}$ in the relevant region, we know that $R_A(z) = R_0(z)F(z)$ fulfills an analogous Hölder estimate, which proves (b).

Regarding condition (c), we know from Example 4.10 that $\|R_A(\lambda \pm i0)\|_{\mathcal{X}_A,\mathcal{X}_A^*} \leq C|\lambda|^{-1/2}$ for large $|\lambda|$, and from part (a) above that $R_A$ is continuous in this norm on $\mathbb{R}\backslash\{0\}$. Hence it only remains to show that $\|R_A(z)\|_{\mathcal{X}_A,\mathcal{X}_A^*}$ is bounded in a neighbourhood of $z = 0$. To that end, recall that the integral kernel of $R_A(z)$ is given by [Yaf10, Ch. 5]

$$R_A(x,x';z) = R_A(x',x;z) = \frac{\theta_1(x,\sqrt{z})\theta_2(x',\sqrt{z})}{\omega(\sqrt{z})} \quad \text{for } x > x', \tag{4.62}$$

where $\theta_{1,2}(x,\zeta)$ with $\operatorname{Im}\zeta \geq 0$ are the solutions of the differential equation $(-\partial_x^2 + w(x) - \zeta^2)\theta_j(x,\zeta) = 0$ with asymptotics $\theta_j(x,\zeta) = e^{\pm ix\zeta} + o(1)$, $\partial_x\theta_j(x,\zeta) = \pm i\zeta e^{\pm ix\zeta}(1 + o(1))$ for $x \to \pm\infty$ (here $+$ for $j = 1$ and $-$ for $j = 2$), and where $\omega$ is the Wronskian of $\theta_1, \theta_2$, a continuous function of $\zeta$, also at $\zeta = 0$. Note that the solutions $\theta_j(x,0)$ are real. In our case, since $v_A(x) \geq 0$, the $\theta_j(x,0)$ must be convex, and not constant as $v_A$ does not vanish identically; hence for $x$ to the right of the support of $v_A$, one has $\theta_1(x,0) = 1$ and $\theta_2(x,0) = cx + d$ with some $c \neq 0$, and the Wronskian $\omega(0)$ does not vanish. Therefore we can choose a neighbourhood $U$ of 0 such that $|\omega(\sqrt{z})| \geq \epsilon > 0$ there.

Now by the Cauchy-Schwarz inequality,

$$\|R_A(z)\|_{\mathcal{X}_A,\mathcal{X}_A^*} = \|\langle x\rangle^{-\alpha}R_A(z)\langle x\rangle^{-\alpha}\|_{\mathcal{H}_A,\mathcal{H}_A} \leq |\omega(\sqrt{z})|^{-1}\prod_{j=1,2}\left(\int dx\,(1+x^2)^{-\alpha}|\theta_j(x,\sqrt{z})|^2\right)^{1/2}. \tag{4.63}$$

Using [DT79, Lemma 2.1(ii)], we can estimate $|\theta_j(x,\zeta)| \leq c(1 + |x|)$ for all $\zeta$. Hence for $z \in U$,

$$\|R_A(z)\|_{\mathcal{X}_A,\mathcal{X}_A^*} \leq \frac{c^2}{\epsilon}\int dx\,(1+x^2)^{-\alpha+1}. \tag{4.64}$$

This integral converges since $\alpha > 3/2$. Hence condition (c) holds.

Property (d) follows directly from the hypothesis on $H_B$, hence Theorem 4.13 is applicable. Further, $R_A(z)$ is compact for $\operatorname{Im} z \neq 0$, and $V$ is bounded in the norm of $\mathfrak{B}(\mathcal{X}^*,\mathcal{X})$ by assumption, hence mutual $\beta$-boundedness of $H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B$ and $H_A \otimes \mathbf{1} + \mathbf{1} \otimes H_B + V$ follows from Corollary 4.14. $\quad\square$

# 5 Application to semibounded operators and quantum inequalities

In this concluding section, we highlight an application of our results that is of interest in the context of quantum physics. We deal with the following question:

Suppose that $A$ is a self-adjoint, unbounded operator on a Hilbert space $\mathcal{H}$, and $B \in \mathfrak{B}(\mathcal{H})$, such that the compression $B^*AB$ is (semi)-bounded. Does this (semi-)boundedness transfer to the compression $BW_\pm^*AW_\pm B$ or – closely related – to $(W_\pm^*BW_\pm)^*A(W_\pm^*BW_\pm)$, where $W_\pm$ is the wave operator of some scattering situation?

We can give a sufficient criterion for this in our context.

**Theorem 5.1.** *Assume that $H_0 \sim_f H_1$ and that $\mathbb{R}\backslash\sigma_{\mathrm{ac}}(H_j)$ are bounded. Denote $W_\pm = W_\pm(H_1,H_0)$. Let $B$ be a bounded operator and $A$ be a self-adjoint (unbounded) operator such that $\|Af(H_j)^{-1}\| < \infty$ for $j = 0, 1$. Then $B^*AB$ is bounded above (below) iff $B^*W_\pm^*AW_\pm B$ is bounded above (below).*

*Proof.* It evidently suffices to show that $W_\pm^*AW_\pm - A$ is bounded. To that end, first note that

$$P_j^{\mathrm{ac}}A - A = (\mathbf{1} - P_j^{\mathrm{ac}})f(H_j)f(H_j)^{-1}A \in \mathfrak{B}(\mathcal{H}), \tag{5.1}$$

since $\mathbf{1} - P_j^{\mathrm{ac}}$ projects onto a bounded subset of the spectrum, and $f(H_j)^{-1}A$ is bounded by assumption. In consequence, also $P_0^{\mathrm{ac}}P_1^{\mathrm{ac}}AP_1^{\mathrm{ac}}P_0^{\mathrm{ac}} - A$ is bounded. Further,

$$W_\pm^*AW_\pm - P_0^{\mathrm{ac}}P_1^{\mathrm{ac}}AP_1^{\mathrm{ac}}P_0^{\mathrm{ac}} = (W_\pm - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}})^*AW_\pm + P_0^{\mathrm{ac}}P_1^{\mathrm{ac}}A(W_\pm - P_1^{\mathrm{ac}}P_0^{\mathrm{ac}}), \tag{5.2}$$

and since $\|W_\pm\| \le 1$, we have

$$\|W_\pm^* A W_\pm - P_0^{ac} P_1^{ac} A P_1^{ac} P_0^{ac}\| \le 2\|f(H_1)(W_\pm - P_1^{ac} P_0^{ac})\| \|Af(H_1)^{-1}\| \tag{5.3}$$

which is finite since $H_0 \sim_f H_1$ and by our assumption on $Af(H_1)^{-1}$. $\qquad\square$

This theoretical result is of interest in quantum physics in the following situation [BCL17].

**Example 5.2** (Quantum mechanical backflow bounds). *In Example 4.10, set $\ell = 2$, $n = 1$, write $P = -i\frac{d}{dx}$, and let $E$ be the spectral projector of $P$ for the interval $[0,\infty)$. Pick a nonnegative Schwartz class function $g$ and set $J(g) := \frac{1}{2}(Pg(X) + g(X)P)$ (the "averaged probability flux operator"). Then $EW_\pm^* J(g)W_\pm E$ is bounded below, but unbounded above.*

*Proof.* It follows from elementary arguments that $EJ(g)E$ is bounded below, but unbounded above [BCL17, Theorem 1]. With $f(\lambda) := (1 + \lambda^2)^{1/4}$, we know from Example 4.10 that $H_0 \sim_f H_1$. Also, $H_0$ has empty discrete spectrum and for $H_1$, the discrete spectrum is bounded. Further, writing $J(f) = f'(X) + 2f(X)P$, it is clear that $J(g)f(H_0)^{-1}$ is bounded, hence also $J(g)f(H_1)^{-1}$ by (B.2). Thus, Theorem 5.1 with $A = J(g)$ and $B = E$ shows that $EW_\pm^* J(g)W_\pm E$ is bounded below but unbounded above. $\qquad\square$

We have thus recovered our results on backflow bounds in [BCL17] for a slightly smaller class of potentials. However, our present methods should allow to generalize the analysis of "quantum inequalities" in scattering situations to backflow of particles with inner degrees of freedom, as well as to a much larger class of semibounded operators relevant in quantum mechanics, e.g., those established in [EFV05].

# A The Privalov-Korn theorem

We note the following variant of the Privalov-Korn theorem on Cauchy integrals, which is adapted to our purposes.

**Lemma A.1.** *Let $\mathcal{E}$ be a Banach space, let $a < b \in \mathbb{R}$, and let $B : [a,b] \to \mathcal{E}$. Suppose there exist a constant $\theta \in (0,1)$ such that*

$$\sup_{\lambda \in [a,b]} \|B(\lambda)\|_{\mathcal{E}} + \sup_{\lambda \neq \lambda' \in [a,b]} \left|\frac{b-a}{\lambda - \lambda'}\right|^\theta \|B(\lambda) - B(\lambda')\|_{\mathcal{E}} =: c < \infty. \tag{A.1}$$

*Then, the function (with the integral defined in the weak sense)*

$$C(\zeta) := \int_a^b \frac{B(\lambda)d\lambda}{\lambda - \zeta}, \quad \zeta \in \mathbb{C}\backslash\mathbb{R}, \tag{A.2}$$

*has limits $C(\lambda \pm i0) := \text{w-}\lim_{\epsilon\downarrow 0} C(\lambda \pm i\epsilon)$ for $\frac{3a+b}{4} =: a' \le \lambda \le b' := \frac{a+3b}{4}$; it is locally Hölder continuous on $[a',b'] \pm [0,\infty)$, and there is a constant $k_\theta > 0$ (depending only on $\theta$, not on $a, b, B, \lambda, \mathcal{E}$) such that*

$$\sup_{\lambda \in [a',b']} \|C(\lambda)\|_{\mathcal{E}} + \sup_{\lambda \neq \lambda' \in [a',b']} \left|\frac{b-a}{\lambda - \lambda'}\right|^\theta \|C(\lambda) - C(\lambda')\|_{\mathcal{E}} \le k_\theta c. \tag{A.3}$$

We will often apply the theorem to $\mathcal{E} = \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ with a Banach space $\mathcal{X}$, where it then also holds with respect to the weak operator topology.

*Proof.* It suffices to prove the statement for $a = 0$, $b = 1$. Once known for that case, it follows for general $a, b$ by considering $\hat{B}(\lambda) := B(a + \lambda(b - a))$, defined for $\lambda \in [0,1]$.

Further, it suffices to prove the statement for $\mathcal{E} = \mathbb{C}$. For general $\mathcal{E}$, choose $\varphi \in \mathcal{E}^*$ and apply the theorem to the $\mathbb{C}$-valued function $\lambda \mapsto \varphi(B(\lambda))$; the resulting integral (A.2) is of the form $\varphi(C(\lambda \pm i0))$ with $C(\lambda \pm i0) \in \mathcal{E}$ by linearity in $\varphi$ and uniformity of the estimate (A.3) in $\|\varphi\|_{\mathcal{E}^*}$.

Now for $a = 0$, $b = 1$, $\mathcal{X} = \mathbb{C}$, we can apply the standard form of the theorem as given, e.g., in [Yaf92, Theorem 1.2.6] and the remark following it. $\qquad\square$

# B    A lemma about differences of operators

We require estimates of differences $h(A) - h(B)$ where $A, B$ are self-adjoint unbounded operators, and $h$ a certain function. The following is a special case of results by Birman, Solomyak and others; see [BS03] for a review.

**Lemma B.1.** *Let $A, B$ be two self-adjoint operators on a common dense domain in a Hilbert space $\mathcal{H}$, such that $B - A$ is bounded. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable such that $h' \in L^p(\mathbb{R})$ for some $p < \infty$, and suppose that $h'$ is (globally) Hölder continuous with some Hölder exponent $\epsilon > 0$. Then $h(B) - h(A)$ is bounded. In particular, for any $\beta \in (0,1)$,*

$$\|(1 + B^2)^{\beta/2} - (1 + A^2)^{\beta/2}\| < \infty. \tag{B.1}$$

*Proof.* The stated conditions on $h$ imply that, in the notation of [BS03], the function $\phi_h(\mu, \lambda) = (h(\mu) - h(\lambda))/(\mu - \lambda)$ falls into the class $\mathfrak{M}$ [BS03, Theorem 8.4]. Therefore, [BS03, Theorem 8.1] is applicable with $\mathfrak{S} = \mathfrak{B}(\mathcal{H})$, yielding that $h(B) - h(A) = Z_h^{A,B}(B - A)$ with a continuous map $Z_h^{A,B} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$.

In particular, these conditions are fulfilled for $h(x) = (1 + x^2)^{\beta/2}$, since $h'(x) = O(|x|^{\beta-1})$ for large $|x|$, and $h''$ is bounded. $\qquad\square$

As a consequence of Lemma B.1, also

$$\|(1 + B^2)^{-\beta/2}(1 + A^2)^{\beta/2}\| < \infty \tag{B.2}$$

by multiplying the bounded operator in (B.1) with $(1 + B^2)^{-\beta/2}$ from the left.

# C    Some kernels of bounded operators

For our purposes, we need norm estimates of certain (singular) integral operators, which we collect here.

**Lemma C.1.** *Let $0 < \gamma < \frac{1}{2}$. Then the (distributional) kernels*

$$K_1(\lambda, \mu) = \frac{(\lambda/\mu)^\gamma}{\lambda - \mu \pm i0}, \qquad K_2(\lambda, \mu) = \frac{(\lambda/\mu)^\gamma}{\lambda + \mu} \tag{C.1}$$

*induce bounded operators $T_1$, $T_2$ on $L^2(\mathbb{R}_+)$.*

*Proof.* Consider the unitary $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$, $(Uf)(x) = e^{x/2}f(e^x)$. The operator $\hat{T}_1 := UT_1U^*$ has the kernel

$$\hat{K}_1(x, y) = e^{x/2}K_1(e^x, e^y)e^{y/2} = \frac{2e^{\gamma(x-y)}}{\sinh(\frac{x-y}{2} \pm i0)}. \tag{C.2}$$

This is a kernel of convolution type, hence we only need to show that the Fourier transform (in the sense of distributions) of

$$f_1(z) = \frac{2e^{\gamma z}}{\sinh(z/2 \pm i0)} \tag{C.3}$$

is a bounded function. This can be extracted from the literature [GR07, Sec. 17.23, formula 20–21], or obtained by comparison with a kernel with the same residue but simpler Fourier transform, e.g., $g(z) = 4i(z + i)^{-1}(z \pm i0)^{-1}$, as $g - f$ is analytic and $L^1$. — Likewise for $K_2$, it suffices to show that

$$f_2(z) = \frac{2e^{\gamma z}}{\cosh(z/2)} \tag{C.4}$$

has a bounded Fourier transform, which is clear since $f_2$ is fast decaying. $\qquad\square$

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