

Polarization-Free Quantum Fields and Interaction

Gandalf Lechner

Institut für Theoretische Physik, Universität Göttingen

37073 Göttingen, Germany

e-mail:lechner@theorie.physik.uni-goettingen.de

April 1, 2007

Abstract

A new approach to the inverse scattering problem proposed by Schroer, is applied to two-dimensional integrable quantum field theories. For any two-particle S-matrix S_2 which is analytic in the physical sheet, quantum fields are constructed which are localizable in wedge-shaped regions of Minkowski space and whose two-particle scattering is described by the given S_2 . These fields are polarization-free in the sense that they create one-particle states from the vacuum without polarization clouds. Thus they provide examples of temperate polarization-free generators in the presence of non-trivial interaction.

1 Introduction

Within the context of collision theory the inverse scattering problem, i.e. the reconstruction of a relativistic quantum field theory from its scattering matrix, is one of a few remaining challenges. This question is important for the construction of field-theoretic models and is addressed in the framework of integrable theories in the so-called form-factor program [1].

It was realized by Schroer [2] that in certain models of this type an intermediate step from the S-matrix to the local quantum fields might be possible. These models admit an algebra of particle creation and annihilation operators satisfying exchange relations involving the two-particle S-matrix [3]. Schroer argued that these operators can be combined to yield Bose-type fields which are localized in wedge-shaped regions and which create single particle states from the vacuum without accompanying polarization clouds (**polarization-free generators**). Because of their relations to the S-matrix, they promised to provide a new tool in the inverse scattering problem in quantum field theory.

A model-independent analysis of these new objects was carried out in [4]. It turned out that polarization-free generators exist in any local theory, but their subtle domain properties render them useless, in general. Assuming for example certain mild

temperateness conditions, which are inevitable if one wants to do scattering theory, it was shown in [4] that the underlying S-matrix must be trivial if the space-time dimension is larger than two. The results obtained for the two-dimensional case indicated that temperate PFGs might well exist in theories with a factorizing S-matrix.

Based on the ideas by Schroer [2, 5], we prove in this article by an explicit construction, cf. also [6], that there are indeed two-dimensional, factorizing theories with non-trivial interaction which allow for temperate PFGs localized in wedge regions. For notational simplicity, we consider only one type of massive, scalar particles. Given a two-particle S-matrix S_2 which has to satisfy well-known physical requirements and additionally analyticity in the physical sheet, we construct temperate polarization-free generators as semi-local Wightman fields which give rise to a wedge-local theory whose two-particle scattering is described by S_2 .

Section 2 introduces Zamolodchikov's algebra which is the basis for our construction and a representation of this algebra on a Hilbert space similar to the bosonic Fock space. In section 3 we define and analyze the wedge-local field. Collision processes of two particles are studied in section 4.

2 The Zamolodchikov algebra

The starting point of our construction is the well-known Zamolodchikov-Faddeev algebra [3] which is commonly used in the context of two-dimensional, integrable models [7]. This algebra is described by quadratic exchange relations involving an operator S_2 which is closely related to the two-particle S-matrix. Since we are dealing with the inverse scattering problem, S_2 is assumed to be given.

We define the Zamolodchikov algebra in a somewhat abstract manner: Let H be a Hilbert space with an antiunitary involution $J_0 = J_0^{-1} = J_0^*$ acting on it and let $S_2 : H \otimes H \rightarrow H \otimes H$ be a unitary operator which fulfills the following exchange relations with J_0 and the "flip operator" $t : H \otimes H \rightarrow H \otimes H$, $t(\psi \otimes \varphi) := \varphi \otimes \psi$:

$$S_2 t = t S_2^*, \quad S_2 J_0^{\otimes 2} = J_0^{\otimes 2} S_2^* . \quad (2.1)$$

Here and in the following $A^{\otimes n}$ denotes the n -fold tensor product of an operator A on H , and $H^{\otimes n}$ is the n -fold tensor product of H . The constraints (2.1) will translate to physically significant properties of the two-particle S-matrix in the representation of the Zamolodchikov algebra given below.

Consider the free $*$ -algebra generated by the symbols 1 , $Z(\psi)$, $Z^\dagger(\psi)$, $\psi \in H$ and $(Z \times Z)(\psi^{(2)})$, $(Z^\dagger \times Z^\dagger)(\psi^{(2)})$, $(Z \times Z^\dagger)(\psi^{(2)})$, $(Z^\dagger \times Z)(\psi^{(2)})$, $\psi^{(2)} \in H^{\otimes 2}$. The Zamolodchikov algebra $\mathcal{Z}(H, J_0, S_2)$ is obtained by imposing several additional relations on this free algebra: All symbols are assumed to depend complex linearly on $\psi \in H$ and $\psi^{(2)} \in H^{\otimes 2}$, respectively. With respect to multiplication in \mathcal{Z} we have 1 as the neutral element and

$$Z^\dagger(\psi)Z(\varphi) = (Z^\dagger \times Z)(\psi \otimes \varphi), \quad \psi, \varphi \in H, \quad (2.2)$$

and analogously for the other symbols. The star operation on \mathcal{Z} is fixed by

$$1^* = 1, \quad Z(\psi)^* = Z^\dagger(J_0\psi), \quad \psi \in H. \quad (2.3)$$

The definition of the algebraic structure of \mathcal{Z} is completed by requiring the relations ($\varphi, \psi \in H$)

$$Z^\dagger(\varphi)Z^\dagger(\psi) = (Z^\dagger \times Z^\dagger)(S_2^*(\psi \otimes \varphi)), \quad (2.4)$$

$$Z(\varphi)Z^\dagger(\psi) = (Z^\dagger \times Z)(S_2(\psi \otimes \varphi)) + \langle J_0\varphi, \psi \rangle \cdot 1. \quad (2.5)$$

Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in H . It should be mentioned that under the assumptions made the exchange relation $Z(\varphi)Z(\psi) = (Z \times Z)(S_2^*(\psi \otimes \varphi))$ holds true as well. All these equalities are consistent because of the properties (2.1) of S_2 .

Having defined the abstract Zamolodchikov algebra, we now turn to the construction of a field theoretic model based on a specific representation of \mathcal{Z} . For the sake of simplicity we consider here only the case of one type of scalar particles of mass m , although our results apply to a more complicated particle spectrum as well.

It is convenient to parametrize the momentum p of a particle by the rapidity θ ,

$$p(\theta) = m \begin{pmatrix} \cosh \theta \\ \sinh \theta \end{pmatrix}. \quad (2.6)$$

We choose as our Hilbert space the rapidity space of a particle of mass m , $H := \mathcal{H}_1 := L^2(\mathbb{R}, d\theta)$. The inversion J_0 is defined as complex conjugation in rapidity space,

$$(J_0\psi)(\theta) := \overline{\psi(\theta)}, \quad \psi \in L^2(\mathbb{R}, d\theta). \quad (2.7)$$

To specify the operator S_2 , we consider a function (denoted by S_2 as well) which is meromorphic on the strip $S(0, \pi) = \{\zeta \in \mathbb{C} : 0 < \text{Im}(\zeta) < \pi\}$ in the complex rapidity plane and continuous on its closure. Furthermore, S_2 should satisfy the following equations for $\theta \in \mathbb{R}$:

$$S_2(\theta)^{-1} = \overline{S_2(\theta)} = S_2(-\theta) = S_2(\theta + i\pi). \quad (2.8)$$

The function S_2 will be called ‘‘scattering function’’ in the following, and the operator S_2 acts by multiplication with it:

$$(S_2\psi^{(2)})(\theta_1, \theta_2) := S_2(\theta_1 - \theta_2) \cdot \psi^{(2)}(\theta_1, \theta_2), \quad \psi^{(2)} \in \mathcal{H}_1^{\otimes 2}. \quad (2.9)$$

The above stated properties of the scattering function imply that S_2 is a unitary operator on $L^2(\mathbb{R}^2)$ which fulfills the consistency requirements (2.1) of the Zamolodchikov algebra. The constraints (2.8) are the well-known properties of unitarity, real analyticity and crossing symmetry (See [8, chapter 3] for a review) which are known to be satisfied by any two-particle S-matrix element obtained from an integrable local quantum field theory on two-dimensional Minkowski space.

Later on we will have to restrict attention to functions S_2 which are bounded on the closed strip $\overline{S(0, \pi)}$ and analytic on its interior, but for the moment the constraints summarized in (2.8) are sufficient to proceed.

After these preparations we can define our representation space; it will be a proper subspace of $\mathcal{F}_{\mathcal{H}_1} := \bigoplus_{n=0}^{\infty} \mathcal{H}_1^{\otimes n}$. The construction is very similar to the construction of the free bosonic or fermionic Fock spaces, the difference being a different symmetrization procedure which depends on the choice of S_2 . In a different context,

this construction has already been carried out in [9].

In the lemma stated below and the subsequent computations we use the following notation: Given an operator A acting on $\mathcal{H}_1 \otimes \mathcal{H}_1$ we denote by $A_n^{i,j}$ the operator which acts on the n -fold tensor product space $\mathcal{H}_1^{\otimes n}$ like A on the i^{th} and j^{th} factor and trivially on the remaining factors, e.g. $A \otimes 1_n = A_{n+2}^{1,2}$ with 1_n denoting the identity on $\mathcal{H}_1^{\otimes n}$.

Furthermore, \mathfrak{S}_n denotes the group of permutations of n elements, and $\tau_i \in \mathfrak{S}_n$, $i = 1, \dots, n-1$, the transposition which exchanges i and $(i+1)$. As in [9], we have the following lemma:

Lemma 1. *Let $R : \mathcal{H}_1^{\otimes 2} \rightarrow \mathcal{H}_1^{\otimes 2}$, $R(\psi \otimes \varphi) := S_2^*(\varphi \otimes \psi)$. The map*

$$\tau_i \longmapsto D_n(\tau_i) := R_n^{i,i+1} \quad (2.10)$$

defines a unitary representation of \mathfrak{S}_n on $\mathcal{H}_1^{\otimes n}$. Moreover, the mean

$$P_n := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} D_n(\pi), \quad (2.11)$$

is an orthogonal projection.

Proof. In order to show that D_n is a representation of \mathfrak{S}_n , one has to check the relations $D_n(\tau_i)^2 = 1$ for $i = 1, \dots, n-1$ and $[D_n(\tau_i), D_n(\tau_j)] = 0$ for $|i-j| > 1$, as well as

$$D_n(\tau_i)D_n(\tau_{i+1})D_n(\tau_i) = D_n(\tau_{i+1})D_n(\tau_i)D_n(\tau_{i+1}) \quad i = 1, \dots, n-2. \quad (2.12)$$

The constraints (2.8) imply the first relation, and the second equality holds because $D_n(\tau_i)$ acts only on the tensor factors i and $(i+1)$. Since S_2^* is a multiplication operator, the different S_2 -factors occurring in the computation of $D_n(\tau_i)D_n(\tau_{i+1})D_n(\tau_i)$ and $D_n(\tau_{i+1})D_n(\tau_i)D_n(\tau_{i+1})$ commute with each other, which implies the third relation.

The unitarity of R follows from (2.8) and carries over to $D_n(\pi)$ since the transpositions generate \mathfrak{S}_n . In particular, the unitarity of the representation D_n implies $P_n = P_n^*$. The equation $P_n^2 = P_n$ holds because \mathfrak{S}_n is a group of $n!$ elements. \square

With the help of the projections P_n we can symmetrize the unsymmetrized Fock space $\mathcal{F}_{\mathcal{H}_1}$ with respect to D_n which yields the definition of our Hilbert space \mathcal{H} :

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_n := P_n \mathcal{H}_1^{\otimes n}, \quad \mathcal{H}_0 := \mathbb{C} \cdot \Omega. \quad (2.13)$$

The projections P_n will be extended to operators on $\mathcal{F}_{\mathcal{H}_1}$ by $P_n \psi := 0$ for $\psi \notin \mathcal{H}_1^{\otimes n}$, and we introduce the notation $P := \bigoplus_{n=0}^{\infty} P_n$.

The zero-particle vector $\Omega = (1, 0, 0, \dots) \in \mathcal{H}$ is interpreted as the physical vacuum. Generalizing the totally symmetric functions known from the free bosonic field, vectors $\psi^{(n)} \in \mathcal{H}_n$ are functions in $L^2(\mathbb{R}^n)$ with the property

$$\psi^{(n)}(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) = (S_2 \psi^{(n)})(\theta_1, \dots, \theta_n) = S_2(\theta_i - \theta_{i+1}) \psi^{(n)}(\theta_1, \dots, \theta_n).$$

As in the free case, we have a representation of the proper orthochronous Poincaré group \mathcal{P}_+^\uparrow on \mathcal{H} . In $1+1$ dimensions \mathcal{P}_+^\uparrow is generated by the translations $x \mapsto x+a$ and the velocity transformations (“boosts”)

$$x \mapsto \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix} x .$$

The Poincaré transformations will be denoted by (a, λ) , where $a \in \mathbb{R}^2$ is a space-time translation and $\lambda \in \mathbb{R}$ the rapidity defining the boost. For $(a, \lambda) \in \mathcal{P}_+^\uparrow$ we set $U_1(a, \lambda) : \mathcal{H}_1 \rightarrow \mathcal{H}_1$,

$$(U_1(a, \lambda)\psi)(\theta) := e^{ip(\theta)a} \cdot \psi(\theta - \lambda) \quad (2.14)$$

and define $U(a, \lambda)$ on $\mathcal{F}_{\mathcal{H}_1}$ via second quantization, $U(a, \lambda) := \bigoplus_{n=0}^\infty U_1(a, \lambda)^{\otimes n}$. We also use the shorter notation $U(a) := U(a, 0)$.

Since the S_2 -factors occurring in the symmetrization P depend only on differences of rapidities and since $U(a)$ and S_2^* are both multiplication operators, P and $U(a, \lambda)$ commute. Thus we conclude that $U(a, \lambda)$ is also a well defined, unitary operator on the subspace $\mathcal{H} \subset \mathcal{F}_{\mathcal{H}_1}$. In view of the spectral properties of $U_1(a, 0)$, U is a positive energy representation of \mathcal{P}_+^\uparrow .

Guided by the Fock representation of the CCR algebra, we now introduce properly symmetrized creation and annihilation operators on \mathcal{H} . We start from their unsymmetrized counterparts $a(\varphi), a^\dagger(\varphi)$, $\varphi \in \mathcal{H}_1$, which are given on $\psi_1 \otimes \dots \otimes \psi_n \in \mathcal{H}_1^{\otimes n}$ by the formulae

$$a(\varphi)\Omega := 0, \quad a(\varphi)\psi_1 \otimes \dots \otimes \psi_n := \sqrt{n} \langle J_0 \varphi, \psi_1 \rangle \psi_2 \otimes \dots \otimes \psi_n, \quad (2.15)$$

$$a^\dagger(\varphi)\Omega := \varphi, \quad a^\dagger(\varphi)\psi_1 \otimes \dots \otimes \psi_n := \sqrt{n+1} \varphi \otimes \psi_1 \otimes \dots \otimes \psi_n. \quad (2.16)$$

These expressions may be extended linearly to unbounded operators defined on the subspace $\mathcal{F}_{\mathcal{H}_1}^{(0)} \subset \mathcal{F}_{\mathcal{H}_1}$ of finite particle number. Contrary to the common definition here $a(\varphi)$ and $a^\dagger(\varphi)$ both depend linearly on φ . Therefore these operators are related by $a(\varphi)^* \supset a^\dagger(J_0 \varphi)$.

The domain of the representation of the Zamolodchikov algebra will be the dense subspace $\mathcal{D} := P\mathcal{F}_{\mathcal{H}_1}^{(0)}$ of symmetrized vectors of finite particle number.

Lemma 2. *The map*

$$Z(\varphi) \mapsto z(\varphi) := Pa(\varphi)P, \quad Z^\dagger(\varphi) \mapsto z^\dagger(\varphi) := Pa^\dagger(\varphi)P \quad (2.17)$$

extends to a representation of the Zamolodchikov algebra $\mathcal{Z}(\mathcal{H}_1, J_0, S_2)$ on \mathcal{D} for which Ω is a cyclic vector. On \mathcal{D} one has

$$z^\dagger(\varphi) = Pa^\dagger(\varphi), \quad z(\varphi) = a(\varphi). \quad (2.18)$$

Proof. Since $\|P\| = 1$, $z(\varphi)$ and $z^\dagger(\varphi)$ are well-defined operators on \mathcal{D} . Obviously $\varphi \mapsto z(\varphi)$, $\varphi \mapsto z^\dagger(\varphi)$ are linear. For the contraction caused by the annihilation operator we will use the notation

$$I_n^{i,j}[\psi_1 \otimes \dots \otimes \psi_n] := \langle J_0 \psi_i, \psi_j \rangle \psi_1 \otimes \dots \otimes \widehat{\psi}_i \otimes \dots \otimes \widehat{\psi}_j \otimes \dots \otimes \psi_n, \quad (2.19)$$

where the hat on $\widehat{\psi}_i$ denotes omission of the corresponding tensor factor. Thus

$$a(\varphi)\psi^{(n)} = \sqrt{n}I_{n+1}^{1,2}[\varphi \otimes \psi^{(n)}], \quad \psi^{(n)} \in \mathcal{H}_1^{\otimes n}.$$

Any symmetrization in the last $(n-1)$ tensor factors of $\psi^{(n)}$ remains unchanged when $a(\varphi)$ is applied, i.e. $a(\varphi)(1_1 \otimes D_{n-1}(\pi))\psi^{(n)} = D_{n-1}(\pi)a(\varphi)\psi^{(n)}$ for any $\pi \in \mathfrak{S}_{n-1}$. This shows that $a(\varphi)\mathcal{H}_n \subset \mathcal{H}_{n-1}$. Therefore the projections P in the definition of $z(\varphi)$ may be omitted: $z(\varphi)\psi = a(\varphi)\psi$ for all $\psi \in \mathcal{D}$. Since $P = 1$ on \mathcal{H} , the equation $z^\dagger(\varphi)\psi = Pa^\dagger(\varphi)\psi$ follows as well.

The $*$ -operation acts like

$$z(\varphi)^* = (Pa(\varphi)P)^* \supset Pa^\dagger(J_0\varphi)P = z^\dagger(J_0\varphi)$$

in agreement with (2.3). To check the relations (2.4, 2.5), let $\psi^{(n)} \in \mathcal{H}_n, \varphi_1, \varphi_2 \in \mathcal{H}_1$. Using $D_{n+2}(\tau_1)^2 = 1$ and $P_{n+2}D_{n+2}(\tau_1) = P_{n+2}$, we get

$$\begin{aligned} z^\dagger(\varphi_1)z^\dagger(\varphi_2)\psi^{(n)} &= \sqrt{(n+1)(n+2)}P_{n+2}(\varphi_1 \otimes P_{n+1}(\varphi_2 \otimes \psi^{(n)})) \\ &= \sqrt{(n+1)(n+2)}P_{n+2}D_{n+2}(\tau_1)(S_2^*(\varphi_2 \otimes \varphi_1) \otimes \psi^{(n)}) \\ &= (z^\dagger \times z^\dagger)(S_2^*(\varphi_2 \otimes \varphi_1))\psi^{(n)}. \end{aligned}$$

Thus the commutation relation (2.4) is verified, and since for $\varphi^{(2)} \in \mathcal{H}_1^{\otimes 2}$ we have $(z^\dagger \times z^\dagger)(\varphi^{(2)})\psi^{(n)} = \sqrt{(n+1)(n+2)}P_{n+2}(\varphi^{(2)} \otimes \psi^{(n)})$ and $(z \times z)(\varphi^{(2)})\psi^n = \sqrt{n(n-1)}I_n^{1,2}I_{n+2}^{2,3}[\varphi^{(2)} \otimes \psi^{(n)}]$, we get the correct multiplication structure (2.2). For the second relation (2.5) we consider the special permutations $\sigma_k \in \mathfrak{S}_n, k = 1, \dots, n$ defined by

$$\sigma_k := \tau_{k-1} \cdots \tau_1, \quad k = 2, \dots, n, \quad \sigma_1 := \text{id}. \quad (2.20)$$

In the following we will denote by D_n^+ the ‘‘bosonic’’ representation of \mathfrak{S}_n which is given as the special case $S_2 \equiv 1$ of D_n . Using $D_n^+(\tau_l)S_n^{i,j} = S_n^{\tau_l(i),\tau_l(j)}D_n^+(\tau_l)$, we compute the unitaries representing σ_k :

$$D_n(\sigma_k) = S_n^{k,k-1}D_n^+(\tau_{k-1}) \cdots S_n^{2,1}D_n^+(\tau_1) = \prod_{j=1}^{k-1} S_n^{k,j}D_n^+(\sigma_k). \quad (2.21)$$

Any permutation $\pi \in \mathfrak{S}_{n+1}$ may be decomposed according to $\pi = \sigma_k\rho$, where $\rho \in \mathfrak{S}_{n+1}$ and $k \in \{1, \dots, n+1\}$ are uniquely determined by the condition $\rho(1) = 1$. For the projection P_{n+1} this entails

$$P_{n+1} = \frac{1}{(n+1)!} \sum_{k=1}^{n+1} \sum_{\rho \in \mathfrak{S}_n} D_{n+1}(\sigma_k)(1_1 \otimes D_n(\rho)) = \frac{1}{n+1} \sum_{k=1}^{n+1} D_{n+1}(\sigma_k)(1_1 \otimes P_n).$$

With this formula at hand we can verify the second commutation relation:

$$\begin{aligned} z(\varphi_1)z^\dagger(\varphi_2)\psi^{(n)} &= (n+1)I_{n+2}^{1,2}[\varphi_1 \otimes P_{n+1}(\varphi_2 \otimes \psi^{(n)})] \\ &= \sum_{k=1}^{n+1} I_{n+2}^{1,2}[\varphi_1 \otimes \prod_{j=1}^{k-1} S_{n+1}^{k,j}D_{n+1}^+(\sigma_k)(\varphi_2 \otimes \psi^{(n)})] \quad (2.22) \end{aligned}$$

The term corresponding to $k = 1$ gives the contribution $\langle J_0 \varphi_1, \varphi_2 \rangle \psi^{(n)}$ as expected from (2.5). We evaluate the remaining sum at $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ and get

$$\sum_{k=1}^n \int d\theta_0 \varphi_1(\theta_0) \varphi_2(\theta_k) \prod_{j=0}^{k-1} S_2(\theta_k - \theta_j) \psi^{(n)}(\theta_0, \theta_1, \dots, \widehat{\theta}_k, \dots, \theta_n), \quad (2.23)$$

where the hat indicates omission of the variable θ_k . We now compute the action of $z^\dagger(\varphi_2)z(\varphi_1)$ on $\psi^{(n)}$:

$$\begin{aligned} z^\dagger(\varphi_2)z(\varphi_1)\psi^{(n)} &= nP_n(\varphi_2 \otimes I_{n+1}^{1,2}[\varphi_1 \otimes \psi^{(n)}]) \\ &= \sum_{k=1}^n \prod_{j=1}^{k-1} S_n^{k,j} D_n^+(\sigma_k) I_{n+2}^{2,3}[\varphi_2 \otimes \varphi_1 \otimes \psi^{(n)}]. \end{aligned} \quad (2.24)$$

Linear and continuous extension of (2.22) in $\varphi_1 \otimes \varphi_2$ and (2.24) in $\varphi_2 \otimes \varphi_1$ yields the operators $(z \times z^\dagger)(\varphi^{(2)})$, $(z^\dagger \times z)(\varphi^{(2)})$, $\varphi^{(2)} \in \mathcal{H}_1^{\otimes 2}$, respectively. In particular we have

$$(z^\dagger \times z)(S_2(\varphi_2 \otimes \varphi_1))\psi^{(n)} = \sum_{k=1}^n \prod_{j=1}^{k-1} S_n^{k,j} D_n^+(\sigma_k) I_{n+2}^{2,3}[S_2(\varphi_2 \otimes \varphi_1) \otimes \psi^{(n)}],$$

and an evaluation at $(\theta_1, \dots, \theta_n)$ yields

$$\begin{aligned} &\sum_{k=1}^n \prod_{j=1}^{k-1} S_2(\theta_k - \theta_j) (I_{n+2}^{2,3}[S_2(\varphi_2 \otimes \varphi_1) \otimes \psi^{(n)}])(\theta_k, \theta_1, \dots, \widehat{\theta}_k, \dots, \theta_n) \\ &= \sum_{k=1}^n \prod_{j=1}^{k-1} S_2(\theta_k - \theta_j) \int d\theta_0 S_2(\theta_k - \theta_0) \varphi_2(\theta_k) \varphi_1(\theta_0) \psi^{(n)}(\theta_0, \theta_1, \dots, \widehat{\theta}_k, \dots, \theta_n) \end{aligned}$$

in agreement with (2.23). So we have established (2.2) and the desired relation $z(\varphi_1)z^\dagger(\varphi_2)\psi = (z^\dagger \times z)(S_2(\varphi_2 \otimes \varphi_1))\psi + \langle J_0 \varphi_1, \varphi_2 \rangle \psi$ for all $\psi \in \mathcal{D}$.

Finally, the cyclicity of Ω follows from $z^\dagger(\psi_1) \cdots z^\dagger(\psi_n)\Omega = \sqrt{n!}P_n(\psi_1 \otimes \dots \otimes \psi_n)$ since P_n is linear and continuous. \square

Usually the Zamolodchikov algebra is described in terms of operator-valued distributions $z(\theta)$, $z^\dagger(\theta)$ [2, 3, 5, 7, 9]. With this notation, our Zamolodchikov operators arise as the formal integrals $z^\dagger(\psi) = \int d\theta \psi(\theta)z^\dagger(\theta)$ and $z(\varphi) = \int d\theta \varphi(\theta)z(\theta)$. In the special cases $S_2 \equiv 1$ and $S_2 \equiv -1$ the Zamolodchikov algebra turns into the CCR and CAR algebra, respectively. Subsequently we will construct bosonic quantum fields for every function S_2 fulfilling (2.8), such that our construction agrees with the usual free field in the case $S_2 \equiv 1$.

3 Wedge-local Fields

With the help of the Zamoldchikov operators, we now define a quantum field Φ on two-dimensional Minkowski space \mathbb{R}^2 . In the following, \mathcal{S} denotes the space of Schwartz test functions.

Definition 1. Let $f \in \mathcal{S}(\mathbb{R}^2)$ and set

$$f^\pm(\theta) := \frac{1}{2\pi} \int d^2x f(\pm x) e^{ip(\theta)x}, \quad p(\theta) = m \begin{pmatrix} \cosh \theta \\ \sinh \theta \end{pmatrix}. \quad (3.1)$$

We regard f^\pm as elements of \mathcal{H}_1 . The field operator $\Phi(f)$ is defined as

$$\Phi(f) := z^\dagger(f^+) + z(f^-). \quad (3.2)$$

In the following proposition we show that Φ shares many properties with the free Bose field, except locality.

Proposition 1. The field operator $\Phi(f)$ has the following properties:

1. $\Phi(f)$ is an unbounded operator defined on \mathcal{D} which leaves this space invariant.
2. For $\psi \in \mathcal{D}$ one has

$$\Phi(f)^*\psi = \Phi(\bar{f})\psi. \quad (3.3)$$

All vectors in \mathcal{D} are entire analytic for $\Phi(f)$. If $f \in \mathcal{S}(\mathbb{R}^2)$ is real, $\Phi(f)$ is essentially self-adjoint on \mathcal{D} .

3. Φ is a solution of the Klein-Gordon equation: For every $f \in \mathcal{S}(\mathbb{R}^2)$, $\psi \in \mathcal{D}$ one has

$$\Phi((\square + m^2)f)\psi = 0. \quad (3.4)$$

4. $\Phi(f)$ transforms covariantly under the representation U of \mathcal{P}_+^\dagger , cf. (2.14):

$$U(g)\Phi(f)U(g)^{-1} = \Phi(f_g), \quad f_g(x) = f(g^{-1}x), \quad g \in \mathcal{P}_+^\dagger. \quad (3.5)$$

5. The vacuum Ω is cyclic for the field Φ : Given any open subset $\mathcal{O} \subset \mathbb{R}^2$, the subspace $D_{\mathcal{O}} := \text{span}\{\Phi(f_1) \cdots \Phi(f_n)\Omega : f_i \in \mathcal{S}(\mathcal{O}), n \in \mathbb{N}_0\}$ is dense in \mathcal{H} .
6. Φ is local if and only if $S_2 \equiv 1$.

Proof. 1) follows directly from the definition of $\Phi(f)$. To establish 2), we note $(f)^\pm = \overline{f^\mp}$, which implies

$$\Phi(f)^*\psi = (z^\dagger(f^+)^* + z(f^-)^*)\psi = (z(\overline{f^+}) + z^\dagger(\overline{f^-}))\psi = \Phi(\bar{f})\psi$$

for $\psi \in \mathcal{D}$. In particular, $\Phi(f)$ is hermitian for real f .

Now let $\psi^{(n)} \in \mathcal{H}_n$ and $N_f := \|f^+\| + \|f^-\|$. From the definitions (2.15,2.16) and $\|P_n\| = 1$ we get the estimates $\|\Phi(f)\psi^{(n)}\| \leq \sqrt{n+1} N_f \|\psi^{(n)}\|$ and

$$\|\Phi(f)^k \psi^{(n)}\| \leq \sqrt{n+k} N_f \|\Phi(f)^{k-1} \psi^{(n)}\| \leq \sqrt{n+k} \cdots \sqrt{n+1} N_f^k \|\psi^{(n)}\|, \quad k \in \mathbb{N}.$$

Thus, for $\zeta \in \mathbb{C}$ we have

$$\sum_{k=0}^{\infty} \frac{|\zeta|^k}{k!} \|\Phi(f)^k \psi^{(n)}\| \leq \|\psi^{(n)}\| \sum_{k=0}^{\infty} \sqrt{\frac{(n+k)!}{n!}} \frac{1}{k!} (|\zeta| N_f)^k < \infty,$$

which shows that every $\psi \in \mathcal{D}$ is an entire analytic vector for $\Phi(f)$. Since \mathcal{D} is dense in \mathcal{H} , we can use Nelson's theorem to conclude that $\Phi(f)$ is essentially selfadjoint on \mathcal{D} . In the following we use the same symbol $\Phi(f)$ for the selfadjoint closure of this operator.

3) follows immediately from $((\square+m^2)f)^\pm = 0$. To prove 4) we choose $\varphi \in \mathcal{H}_1, \psi^{(n)} \in \mathcal{H}_n, g \in \mathcal{P}_+^\dagger$ and calculate

$$\begin{aligned} U(g)z^\dagger(\varphi)U(g)^*\psi^{(n)} &= \sqrt{n+1} U(g)P_{n+1}(\varphi \otimes U(g)^*\psi^{(n)}) \\ &= \sqrt{n+1} P_{n+1}(U(g)\varphi \otimes \psi^{(n)}) \\ &= z^\dagger(U(g)\varphi)\psi^{(n)}. \end{aligned}$$

This implies $U(g)z(\varphi)U(g)^* = z(J_0U(g)J_0\varphi)$. Now let $g = (a, \lambda)$. Because of $J_0U(a, \lambda)J_0 = U(-a, \lambda)$ one gets $U(g)\Phi(f)U(g)^* = z^\dagger(U(a, \lambda)f^+) + z(U(-a, \lambda)f^-)$, and the latter expression is equal to $\Phi(f_{(a, \lambda)})$ since $f_{(a, \lambda)}^\pm = U(\pm a, \lambda)f^\pm$.

5) Let $\mathcal{P}(\mathcal{O})$ denote the algebra generated by all polynomials in the field $\Phi(f)$ with $f \in \mathcal{S}(\mathcal{O})$. By standard analyticity arguments making use of the spectrum condition it follows that $\mathcal{P}(\mathcal{O})\Omega$ is dense in \mathcal{H} if and only if $\mathcal{P}(\mathbb{R}^2)\Omega$ is dense in \mathcal{H} . Choosing $f \in \mathcal{S}(\mathbb{R}^2)$ such that $f^- = 0$, we conclude that $z^\dagger(f^+) \in \mathcal{P}(\mathbb{R}^2)$ which implies that $\mathcal{P}(\mathbb{R}^2)\Omega$ is dense in \mathcal{H} .

6) One cannot expect the field $\Phi(f)$ to be local since it creates one-particle states from the vacuum ($\Phi(f)\Omega = f^+ \in \mathcal{H}_1$), a property which is known to be compatible with locality only in the free case, as the Jost-Schroer-Theorem states [10]. Explicitly one may look at the two particle contribution of the field commutator applied to the vacuum:

$$\begin{aligned} P_2[\Phi(f), \Phi(g)]\Omega &= \sqrt{2}P_2(f^+ \otimes g^+ - g^+ \otimes f^+) \\ &= 2^{-1/2}(f^+ \otimes g^+ - g^+ \otimes f^+ - S_2^*(f^+ \otimes g^+ - g^+ \otimes f^+)). \end{aligned}$$

If $S_2 \equiv 1$ this expression vanishes, as expected from the locality of the free field. However, Φ is not local in the general case. \square

Remark: It is worth mentioning that $\Phi(f)$ fulfills the temperateness assumption with respect to Poincaré transformations made in [4]. In fact,

$$\|\Phi(f)U(g)\psi^{(n)}\| \leq \sqrt{n+1}N_f\|U(g)\psi^{(n)}\| = \sqrt{n+1}N_f\|\psi^{(n)}\| \quad (3.6)$$

for any $f \in \mathcal{S}(\mathbb{R}^2), g \in \mathcal{P}_+^\dagger, \psi^{(n)} \in \mathcal{H}_n$.

Since locality is one of the fundamental principles in relativistic quantum field theory, the non-local character of our field Φ seems to be a severe problem. But, as we will show below, the field is not completely delocalized. It is localizable in certain

unbounded wedge regions of Minkowski space which are defined as follows. A subset of \mathbb{R}^2 will be called wedge if it is a Poincaré transform of the right hand wedge

$$W_R := \{x \in \mathbb{R}^2 : x_1 > |x_0|\}. \quad (3.7)$$

The wedge W_R and its causal complement $W_L = -W_R$ will be of special significance. Since the boost transformations leave W_R invariant, the set of all wedges \mathcal{W} is given by

$$\mathcal{W} = \bigcup_{x \in \mathbb{R}^2} \{W_R + x, W_L + x\}. \quad (3.8)$$

Starting from Φ , we will exhibit for every wedge $W \in \mathcal{W}$ quantum fields localized in W . These fields transform covariantly under the given representation U of the Poincaré group, have the vacuum as a cyclic vector and commute if they are localized in space-like separated wedges, i.e. they are of Bose type.

We assign the field $\Phi(f)$ to the left wedge $W_L + x$ whenever $\text{supp}(f) \subset W_L + x$. In view of (3.5) this assignment is covariant with regard to the automorphic action of \mathcal{P}_+^\uparrow .

The crucial step is to exhibit a covariant field Φ' such that $\Phi'(f)$ is localized in $W_R + y$ if $\text{supp}(f) \subset W_R + y$, and which is local relative to Φ . Schroer suggested [5] that Φ and Φ' are related by the adjoint action of the operator

$$J := \bigoplus_{n=0}^{\infty} S_n J_0^{\otimes n} \quad \text{where} \quad S_n := \prod_{1 \leq i < j \leq n} S_n^{j,i}. \quad (3.9)$$

The properties of J are specified in the following lemma.

Lemma 3. 1. J is an antiunitary involution on \mathcal{H} , i.e. $J = J^{-1} = J^*$. It acts on \mathcal{H}_n according to

$$JP_n = P_n D_n^+(\iota_n) J_0^{\otimes n}, \quad (3.10)$$

where $\iota_n \in \mathfrak{S}_n$ denotes the permutation $(1, \dots, n) \mapsto (n, \dots, 1)$.

2. J represents the space-time reflection $j : x \mapsto -x$, i.e. for $g \in \mathcal{P}_+^\uparrow$ we have

$$JU(g)J = U(jgj). \quad (3.11)$$

Proof. 1) The (anti-) unitarity of S_n , $J_0^{\otimes n}$ and (2.1) imply that $J_n := S_n J_0^{\otimes n}$ satisfies $J_n = J_n^{-1} = J_n^*$ which entails the same properties for J . Thus J is an antiunitary involution on $\mathcal{F}_{\mathcal{H}_1}$. To prove (3.10), we first note that the equality $\iota_n = \sigma_2 \cdots \sigma_n$ involving the special permutations σ_k defined in (2.20) holds. Next we want to show $D_n(\iota_n) = S_n D_n^+(\iota_n)$ by induction in n , the case $n = 1$ being trivial. The step from n to $n + 1$ is achieved with the help of (2.21):

$$\begin{aligned} D_{n+1}(\iota_{n+1}) &= D_{n+1}(\iota_n \sigma_{n+1}) = (S_n D_n^+(\iota_n) \otimes 1_1) \prod_{j=1}^n S_{n+1}^{n+1,j} D_{n+1}^+(\sigma_{n+1}) \\ &= (S_n \otimes 1_1) \prod_{j=1}^n S_{n+1}^{n+1,n+1-j} (D_n^+(\iota_n) \otimes 1_1) D_{n+1}^+(\sigma_{n+1}) \\ &= S_{n+1} D_{n+1}^+(\iota_{n+1}). \end{aligned}$$

Since $D_n(\iota_n) = 1$ on \mathcal{H}_n , this implies that J acts on \mathcal{H}_n like $J_n = S_n J_0^{\otimes n} = J_0^{\otimes n} S_n^* = J_0^{\otimes n} D_n^+(\iota_n)$. It remains to show that $J_0^{\otimes n} D_n^+(\iota_n)$ commutes with P_n . The equations $\iota_n \tau_i \iota_n = \tau_{n-i}$ and $\iota_n^2 = \text{id}$ hold in \mathfrak{S}_n and lead to

$$\begin{aligned} J_0^{\otimes n} D_n^+(\iota_n) D_n(\tau_i) &= J_0^{\otimes n} D_n^+(\iota_n) S_n^{i+1,i} D_n^+(\tau_i) \\ &= J_0^{\otimes n} S_n^{n-i, n+1-i} D_n^+(\iota_n \tau_i) \\ &= S_n^{n-i+1, n-i} J_0^{\otimes n} D_n^+(\tau_{n-i} \iota_n) \\ &= D_n(\tau_{n-i}) J_0^{\otimes n} D_n^+(\iota_n). \end{aligned}$$

Considering a general permutation $\pi = \tau_{i_1} \cdots \tau_{i_k}$, this implies that J acts on the representation of \mathfrak{S}_n via $J_0^{\otimes n} D_n^+(\iota_n) D_n(\pi) = D_n(\pi') J_0^{\otimes n} D_n^+(\iota_n)$, $\pi' := \iota_n \pi \iota_n$. Hence the mean over the group, the symmetrization P_n , is left invariant. So we conclude that (3.10) holds and J can be restricted to \mathcal{H} .

2) To prove (3.11), we notice that $JU_1(a, \lambda)J = U_1(-a, \lambda) = U_1(j(a, \lambda)j)$ holds on the one-particle space for any $(a, \lambda) \in \mathcal{P}_+^1$. But in view of (3.10), this equality holds on \mathcal{H}_n and thus on \mathcal{H} , too. \square

Using this TCP operator J , we define the ‘‘reflected’’ Zamolodchikov operators

$$z^\dagger(\varphi)' := Jz^\dagger(\varphi)J, \quad z(\varphi)' := Jz(\varphi)J, \quad (3.12)$$

and a ‘‘reflected’’ field

$$\Phi'(f) := J\Phi(f^j)J, \quad f^j(x) := \overline{f(-x)}. \quad (3.13)$$

These are linear, unbounded operators defined on \mathcal{D} . The field $\Phi'(f)$ has the same properties as $\Phi(f)$: It depends linearly on f , is essentially self-adjoint on \mathcal{D} if f is real and provides a solution of the Klein-Gordon equation (Note $\Phi'(f)\Omega = f^+ = \Phi(f)\Omega$). $\Phi'(f)$ transforms covariantly under the representation U of \mathcal{P}_+^1 . In view of $J^2 = 1$ and $J\Omega = \Omega$ we have

$$\Phi'(f_1) \cdots \Phi'(f_n)\Omega = J\Phi(f_1^j) \cdots \Phi(f_n^j)\Omega$$

and conclude that Ω is cyclic for Φ' as well. Furthermore, $\Phi'(f)$ will turn out to be localized in $W_R + \text{supp}(f)$. As a prerequisite for this important result, we compute the commutation relations of creation and annihilation operators with their reflected counterparts in the following lemma.

Lemma 4. *Let $\varphi_1, \varphi_2 \in \mathcal{H}_1$. The following commutation relations hold on \mathcal{D} :*

$$[z^\dagger(\varphi_1)', z^\dagger(\varphi_2)] = 0, \quad [z(\varphi_1)', z(\varphi_2)] = 0. \quad (3.14)$$

The ‘‘mixed’’ commutators act on \mathcal{H}_n , $n \in \mathbb{N}_0$, as multiplication operators ($\underline{\theta} = (\theta_1, \dots, \theta_n)$):

$$\begin{aligned} [z^\dagger(\varphi_1)', z(\varphi_2)]\psi^{(n)} &= B_n^{\varphi_1, \varphi_2} \cdot \psi^{(n)}, \quad B_n^{\varphi_1, \varphi_2}(\underline{\theta}) = - \int d\theta \overline{\varphi_1(\theta)} \varphi_2(\theta) \prod_{j=1}^n S_2(\theta_j - \theta), \\ [z(\varphi_1)', z^\dagger(\varphi_2)]\psi^{(n)} &= C_n^{\varphi_1, \varphi_2} \cdot \psi^{(n)}, \quad C_n^{\varphi_1, \varphi_2}(\underline{\theta}) = + \int d\theta \overline{\varphi_1(\theta)} \varphi_2(\theta) \prod_{j=1}^n S_2(\theta - \theta_j). \end{aligned}$$

Proof. With the help of the preceding lemma we can compute the action of $z^\dagger(\varphi)'$ and $z(\varphi)'$, $\varphi \in \mathcal{H}_1$, on \mathcal{H}_n :

$$\begin{aligned} z^\dagger(\varphi)'\psi^{(n)} &= \sqrt{n+1}JP_{n+1}(\varphi \otimes J\psi^{(n)}) = \sqrt{n+1}P_{n+1}(\psi^{(n)} \otimes J_0\varphi), \\ z(\varphi)'\psi^{(n)} &= \sqrt{n}JI_{n+1}^{1,2}[\varphi \otimes J\psi^{(n)}] = \sqrt{n}I_{n+1}^{1,n+1}[J_0\varphi \otimes \psi^{(n)}]. \end{aligned}$$

The creation operators $z^\dagger(\varphi_1)$ commute with $z^\dagger(\varphi_2)'$ on \mathcal{H}_n because these operators perform the independent operations of tensorial multiplication from the left and right, respectively. So $[z^\dagger(\varphi_1)', z^\dagger(\varphi_2)]\psi = 0$ for all $\varphi_1, \varphi_2 \in \mathcal{H}_1$, $\psi \in \mathcal{D}$, and a similar argument leads to $[z(\varphi_1)', z(\varphi_2)]\psi = 0$.

It remains to investigate the mixed commutation relations. We use a decomposition of permutations $\pi \in \mathfrak{S}_{n+1}$ similar to the one used in the proof of Lemma 2, namely $D_{n+1}(\pi) = D_{n+1}(\sigma'_k)(D_n(\rho) \otimes 1_1)$. (In the following the notation $\varphi^* := J_0\varphi$ for vectors $\varphi \in \mathcal{H}_1$ is used.)

$$\begin{aligned} [z^\dagger(\varphi_1)', z(\varphi_2)]\psi^{(n)} &= \sqrt{n}P_n(z(\varphi_2)\psi^{(n)} \otimes \varphi_1^*) - \sqrt{n+1}I_{n+2}^{1,2}[\varphi_2 \otimes z^\dagger(\varphi_1)'\psi^{(n)}] \\ &= nP_nI_{n+2}^{1,2}[\varphi_2 \otimes \psi^{(n)} \otimes \varphi_1^*] - (n+1)I_{n+2}^{1,2}[\varphi_2 \otimes P_{n+1}(\psi^{(n)} \otimes \varphi_1^*)] \\ &= nI_{n+2}^{1,2}[(1_2 \otimes P_n)(\varphi_2 \otimes \psi^{(n)} \otimes \varphi_1^*)] - I_{n+2}^{1,2}[\varphi_2 \otimes \sum_{k=1}^{n+1} D_{n+1}(\sigma'_k)(\psi^{(n)} \otimes \varphi_1^*)] \end{aligned}$$

This expression vanishes apart from the term corresponding to $k = n+1$, as can be seen using $D_{n+1}(\sigma'_k) = 1_1 \otimes D_n(\sigma'_k)$, $k < n+1$, and the decomposition $P_n = \frac{1}{n} \sum_{r=1}^n D_n(\sigma'_r)(P_{n-1} \otimes 1_1)$. Taking into account $\sigma'_{n+1} = \tau_1 \cdots \tau_n$ and

$$D_{n+1}(\sigma'_{n+1}) = S_{n+1}^{2,1}D_{n+1}^+(\tau_1) \cdots S_{n+1}^{n+1,n}D_{n+1}^+(\tau_n) = \prod_{j=2}^{n+1} S_{n+1}^{j,1}D_{n+1}^+(\sigma'_{n+1}),$$

we have thus shown

$$\begin{aligned} [z^\dagger(\varphi_1)', z(\varphi_2)]\psi^{(n)} &= -I_{n+2}^{1,2} \left[\varphi_2 \otimes D_{n+1}(\sigma'_{n+1})(\psi^{(n)} \otimes \varphi_1^*) \right] \\ &= -I_{n+2}^{1,2} \left[\varphi_2 \otimes \prod_{j=1}^n S_{n+1}^{1+j,1} \varphi_1^* \otimes \psi^{(n)} \right]. \end{aligned}$$

So this commutator acts on \mathcal{H}_n by multiplication with the function $B_n^{\varphi_1, \varphi_2}$. As

$$[z(\varphi_1)', z^\dagger(\varphi_2)]\psi^{(n)} = -[z^\dagger(\varphi_1^*)', z(\varphi_2^*)]^* \psi^{(n)} = -\overline{B_n^{\varphi_1^*, \varphi_2^*}} \cdot \psi^{(n)},$$

and $\overline{S_2(\theta_j - \theta)} = S_2(\theta - \theta_j)$, we also obtain $C_n^{\varphi_1, \varphi_2} = -\overline{B_n^{\varphi_1^*, \varphi_2^*}}$ as claimed. \square

Let us now assume that S_2 is analytic on the open strip $S(0, \pi)$, and bounded on its closure. Since poles in the scattering function indicate bound states [7], we exclude hereby the possibility of bound states in agreement with the formulation of the model describing a single species of particles.

We will show now that the field $\Phi'(f)$ is localized in $W_R + \text{supp}(f)$. Since we assigned $\Phi(g)$ to the wedge $W_L + x$ if $\text{supp}(g) \subset W_L + x$, we have to show that $\Phi'(f)$ and $\Phi(g)$

commute in the sense that $[\Phi'(f), \Phi(g)]$ vanishes on \mathcal{D} if $W_L + \text{supp}(g)$ is spacelike to $W_R + \text{supp}(f)$. This is the case if and only if $\text{supp}(f) - \text{supp}(g) \subset W_R$. Using the translation covariance of Φ and Φ' , it suffices to consider the case $\text{supp}(f) \subset W_R$, $\text{supp}(g) \subset W_L$ only.

Let $f \in \mathcal{S}(W_R)$, $g \in \mathcal{S}(W_L)$. Since g has support in the cone W_L , its Fourier transform \tilde{g} is the boundary value of a function analytic in the tube $\mathbb{R}^2 - iW_L$ [11]. Using the rapidity parametrization (3.1) one calculates that

$$\text{Im}(p(\theta + i\mu)) = m \sin \mu \begin{pmatrix} \sinh \theta \\ \cosh \theta \end{pmatrix} \in -W_L \text{ for } 0 < \mu < \pi .$$

Thus g^+ has an analytic continuation to $S(0, \pi)$. Since $p(\theta + i\pi) = -p(\theta)$, the value at the upper boundary of the strip is given by $\overline{g^+}(\theta + i\pi) = g^-(\theta)$. Considering W_R instead of W_L one concludes that g^- as well as $\overline{g^+}$ are analytic on the strip $S(-\pi, 0)$. The fact that g is a Schwartz function and $|e^{ip(\theta)x}| = e^{-\text{Im}(p(\theta)) \cdot x}$, $\text{Im}(p(\theta))x \geq 0$ for $\text{Im}(\theta) \in [0, \pi]$ and $x \in W_L$, imply that $g^+(\theta)$, $\text{Im}(\theta) \in (0, \pi)$ decays exponentially to zero for $\text{Re}(\theta) \rightarrow \pm\infty$. On the boundary we have $g^+(\theta) \rightarrow 0$, $g^-(\theta) = \overline{g^+}(\theta + i\pi) \rightarrow 0$ for $\theta \rightarrow \pm\infty$ since g is a Schwartz space test function.

All these considerations apply to f^j as well since $f^j \in \mathcal{S}(W_L)$. In view of $(f^j)^\pm = \overline{f^\pm}$ we have analyticity for f^+ and f^- in $S(-\pi, 0)$ and $S(0, \pi)$, respectively, as well as exponential decay for $\text{Re}(\theta) \rightarrow \pm\infty$ in these strips.

Having recalled these analyticity properties we now want to prove that $\Phi'(f)$ and $\Phi(g)$ commute on \mathcal{H}_n . In view of the preceding lemma the commutator simplifies to $[\Phi'(f), \Phi(g)] = [z^\dagger(\overline{f^+})', z(g^-)] + [z(\overline{f^-})', z^\dagger(g^+)]$ and it remains to show that $B_n^{\overline{f^+}, g^-} + C_n^{\overline{f^-}, g^+} = 0$. The integrand of $C_n^{\overline{f^-}, g^+}$ is analytic on $S(0, \pi)$ and of fast decrease for $\text{Re}(\theta) \rightarrow \pm\infty$ because the test functions decay exponentially and S_2 is bounded on $S(0, \pi)$. This enables us to shift the integration from \mathbb{R} to $\mathbb{R} + i\pi$:

$$\begin{aligned} C_n^{\overline{f^-}, g^+}(\theta_1, \dots, \theta_n) &= \int d\theta f^-(\theta) g^+(\theta) \prod_{j=1}^n S_2(\theta - \theta_j) \\ &= \int d\theta f^-(\theta + i\pi) g^+(\theta + i\pi) \prod_{j=1}^n S_2(\theta + i\pi - \theta_j) \\ &= \int d\theta f^+(\theta) g^-(\theta) \prod_{j=1}^n S_2(\theta_j - \theta) \\ &= -B_n^{\overline{f^+}, g^-}(\theta_1, \dots, \theta_n) , \end{aligned}$$

where in the third equality we used the crossing symmetry $S_2(\theta + i\pi) = S_2(-\theta)$. Hence $[\Phi'(f), \Phi(g)]$ vanishes on \mathcal{D} if S_2 is analytic in $S(0, \pi)$.

Since we have the vacuum as an analytic vector for $\Phi'(f)$ and $\Phi(g)$, we may apply an argument of Borchers and Zimmermann [12] to conclude that the unitary groups $e^{it\Phi'(f)}$ and $e^{is\Phi(g)}$, $f, g \in \mathcal{S}(W_R)$ real, $t, s \in \mathbb{R}$, commute as well. Thus one can proceed from the fields to nets of local von Neumann algebras affiliated with wedges. We summarize our results in the following proposition:

Proposition 2. *Let S_2 be a bounded, continuous function on $\overline{S(0, \pi)}$ which is analytic on the open strip $S(0, \pi)$ and satisfies the constraints (2.8).*

1. *The field Φ' has the properties 1)-6) stated in Proposition 1 for the field Φ .*
2. *Let $f, g \in \mathcal{S}(\mathbb{R}^2)$. If $W_R + \text{supp}(f)$ and $W_L + \text{supp}(g)$ lie spacelike to each other, $\Phi'(f)$ and $\Phi(g)$ commute.*

The assignment of $\Phi'(f)$ to the wedge $W_L + x$ whenever $\text{supp}(f) \subset W_L + x$ thus completes the definition of our wedge-local quantum field theory.

It remains to be shown that the scattering of our model is given by S_2 . This will be done in the next section. Due to the weak localization properties of Φ and Φ' , we have to restrict our analysis to collision processes of two particles.

4 Scattering States

For the analysis of scattering processes we employ the Haag-Ruelle scattering theory in a form used in [4] for wedge-localized fields. We briefly recall how two-particle scattering states can be constructed in this case.

We first introduce some notation: The velocity support of a test function f will be denoted by $\Gamma(f) := \{(1, p_1/\omega_p) \in \mathbb{R}^2 : p \in \text{supp} f\}$, where $\omega_p = \sqrt{p_1^2 + m^2}$, and we write $\Gamma(g) \prec \Gamma(f)$ if $\Gamma(f) - \Gamma(g) \subset W_R$. One verifies that $\Gamma(g) \prec \Gamma(f)$ implies that the support of g^+ lies left to the support of f^+ , i.e. $\text{supp}(f^+) - \text{supp}(g^+) \subset \mathbb{R}^+$. Furthermore, we use the family of functions depending on the time parameter t

$$f_t(x) := \frac{1}{2\pi} \int d^2p \tilde{f}(p) e^{i(p_0 - \omega_p)t} e^{-ipx}. \quad (4.1)$$

It is well-known that the support of the function f_t is essentially localized in $t\Gamma(f)$ for large $|t|$.

To construct outgoing scattering states, we consider test functions f and g with $\Gamma(g) \prec \Gamma(f)$ and whose energy-momentum supports are contained in a neighbourhood of the mass shell $H_m^+ = \{(p_1, \sqrt{p_1^2 + m^2}) : p_1 \in \mathbb{R}^2\}$. Thus at very late or early times, the operators $\Phi'(f)$ and $\Phi(g)$ are essentially localized in $W_R + t\Gamma(f)$ and $W_L + t\Gamma(g)$, respectively [4]. Since $\Gamma(g) \prec \Gamma(f)$, these localization regions are spacelike separated for large positive t and their distance increases linearly with t . In view of $\Phi(g)\Omega = g^+$, $\Phi'(f)\Omega = f^+$, one can therefore define the outgoing two-particle state

$$\lim_{t \rightarrow \infty} \Phi(g_t) \Phi'(f_t) \Omega = (g^+ \times f^+)_{\text{out}}, \quad (4.2)$$

and similarly

$$\lim_{t \rightarrow \infty} \Phi'(f_t) \Phi(g_t) \Omega = (f^+ \times g^+)_{\text{out}}. \quad (4.3)$$

Since the operators $\Phi'(f_t)$ and $\Phi(g_t)$ commute for $t \rightarrow \infty$, we have symmetric scattering states $(g^+ \times f^+)_{\text{out}} = (f^+ \times g^+)_{\text{out}}$ as required for a Boson.

To construct incoming scattering states one has to exchange f and g because then

$W_L + t\Gamma(f)$ and $W_R + t\Gamma(g)$ are far apart and spacelike separated in the limit $t \rightarrow -\infty$. Hence

$$\lim_{t \rightarrow -\infty} \Phi(f_t)\Phi'(g_t)\Omega = (f^+ \times g^+)_{\text{in}} = (g^+ \times f^+)_{\text{in}}. \quad (4.4)$$

In our model these limits can be easily computed: The definition (4.1) implies $f_t^+ = f^+$, $g_t^+ = g^+$ and furthermore we have $f_t^- = 0$, $g_t^- = 0$ since the supports of \tilde{f} , \tilde{g} do not intersect the lower mass shell $-H_m^+$.

So the time-dependence drops out and we arrive at the following form of the two-particle scattering states:

$$\begin{aligned} (g^+ \times f^+)_{\text{out}} &= \lim_{t \rightarrow \infty} \Phi(g_t)\Phi'(f_t)\Omega = z^\dagger(g^+)f^+ = \sqrt{2}P_2(g^+ \otimes f^+), \\ (g^+ \times f^+)_{\text{in}} &= \lim_{t \rightarrow -\infty} \Phi(f_t)\Phi'(g_t)\Omega = z^\dagger(f^+)g^+ = \sqrt{2}P_2(f^+ \otimes g^+). \end{aligned}$$

Varying f, g within the above limitations, we obtain a total set of two-particle scattering states.

In an integrable quantum field theory in two dimensions, the two-particle S-matrix element ${}_{\text{out}}\langle \theta_1, \theta_2 | \theta_1, \theta_2 \rangle_{\text{in}} = S_2(|\theta_1 - \theta_2|)$ depends on the absolute value of the difference of the incoming rapidities only and allows for a meromorphic continuation when restricted to $\theta_1 - \theta_2 > 0$. This meromorphic function is the scattering function S_2 we have worked with so far. The two-particle S-matrix, however, acts on \mathcal{H}_2 as the multiplication operator

$$(S\psi)(\theta_1, \theta_2) := S_2(|\theta_1 - \theta_2|)\psi(\theta_1, \theta_2).$$

Since $\text{supp}(f^+) - \text{supp}(g^+) \subset \mathbb{R}_+$ and $S_2(-\theta) = S_2(\theta)^{-1}$, we have

$$S(f^+ \otimes g^+) = S_2(f^+ \otimes g^+), \quad S(g^+ \otimes f^+) = S_2^{-1}(g^+ \otimes f^+)$$

and

$$\begin{aligned} S(g^+ \times f^+)_{\text{out}} &= 2^{-1/2}S(g^+ \otimes f^+ + S_2^*(f^+ \otimes g^+)) \\ &= 2^{-1/2}(S_2^{-1}(g^+ \otimes f^+) + f^+ \otimes g^+) \\ &= \sqrt{2}P_2(f^+ \otimes g^+) \\ &= (g^+ \times f^+)_{\text{in}}. \end{aligned}$$

Having verified the action of S on a total set of scattering states, it follows by linearity and continuity that the two-particle S-matrix of our model is the multiplication operator S corresponding to the scattering function S_2 .

We have shown that to many factorizing S-matrices a wedge-local theory with the correct scattering behaviour can be constructed with the help of polarization-free generators. Of course the important question arises whether there exist local fields in our model and how they are related to the S-matrix. As Schroer pointed out, this question amounts to the analysis of intersections of certain operator algebras [5] which is another crucial step in the approach to the inverse scattering problem presented here.

Acknowledgements

I would like to thank D. Buchholz for many discussions which have been essential for the work. Special thanks are also due to J. Mund for helpful conversations and valuable advice. Financial support by the Lucia-Pfohe Stiftung is gratefully acknowledged.

References

- [1] H. Babujian, A. Fring, M. Karowski, A. Zapletal: *Exact form factors in integrable quantum field theories: The Sine-Gordon model*, Nucl. Phys. **B538** (1999) 535-586 [arXiv:hep-th/9805185]
- [2] B. Schroer: *Modular wedge localization and the $d = 1+1$ formfactor program*, Annals Phys. **275** (1999) 190-223 [arXiv:hep-th/9712124].
- [3] A. Zamolodchikov, A. Zamolodchikov : *Factorized S-Matrices as the Exact Solutions of Certain Relativistic Quantum Field Theory Models*, Annals Phys. **120** (1979) 253-291
- [4] H.-J. Borchers, D. Buchholz, B. Schroer: *Polarization-Free Generators and the S-Matrix*, Commun. Math. Phys. **219** (2001) 125-140 [arXiv: hep-th/0003243]
- [5] B. Schroer, H. W. Wiesbrock: *Modular Constructions of Quantum Field Theories with Interactions*, Rev. Math. Phys. **12** (2000) 301-326 [arXiv:hep-th/9812251]
- [6] G. Lechner: *Polarisationsfreie Generatoren in zwei Dimensionen*, diploma thesis, Uni Göttingen 2002 (unpublished)
- [7] E. Abdalla, M. C. Abdalla, K. D. Rothe: *Non-Perturbative Methods in 2 Dimensional Quantum Field Theory*, Chapter VIII, World Scientific 1991
- [8] C. Korff: *Lie algebraic structures in integrable models, affine Toda field theory*, PhD thesis, FU Berlin 2000 [arXiv: hep-th/0008200]
- [9] A. Liguori, M. Mintchev: *Fock Spaces with Generalized Statistics*, Lett. Math. Phys. **33** (1995) 283-295 [arXiv: hep-th/9403039]
- [10] R. F. Streater, A. S. Wightman: *PCT, Spin and Statistics, and All That*, W.A. Benjamin, New York, 1964
- [11] M. Reed, B. Simon: *Methods in Modern Mathematical Physics II*, Academic Press 1975
- [12] H.-J. Borchers, W. Zimmermann: *On the Self-Adjointness of Field Operators*, Nuovo Cimento **31** (1963) 1047-1059.