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# Modular Nuclearity and Localization

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**Abstract**

Within the algebraic setting of quantum field theory, a condition is given which implies that the intersection of algebras generated by field operators localized in wedge–shaped regions of two–dimensional Minkowski space is non–trivial; in particular, there exist compactly localized operators in such theories which can be interpreted as local observables. The condition is based on spectral (nuclearity) properties of the modular operators affiliated with wedge algebras and the vacuum state and is of interest in the algebraic approach to the formfactor program, initiated by Schroer. It is illustrated here in a simple class of examples.

## 1 Introduction

There is growing evidence that algebraic quantum field theory [23] not only is useful in structural analysis but provides also a framework for the construction of models. Basic ingredients in this context are, on the one hand, the algebras affiliated with wedge shaped regions in Minkowski space, called wedge algebras for short. On the other hand there enter the modular groups corresponding to these algebras and the vacuum state by Tomita–Takesaki theory.

The wedge algebras are distinguished by the fact that the associated modular groups can be interpreted as unitary representations of specific Poincaré transformations. This fact was established first by Bisognano and Wichmann in the Wightman framework of quantum field theory [4] and, more recently, by Borchers in the algebraic setting [6], cf. also [22, 32]. It triggered attempts to construct families of such algebras directly within the algebraic framework [9, 31].

A particularly interesting development was initiated by Schroer [35] who, starting from a given factorizing scattering matrix in two spacetime dimensions, recognized how one may reconstruct from these data a family of wedge algebras satisfying locality. A complete construction of these algebras for a simple class of scattering matrices was given in [27]. These results are a first important step in an algebraic approach to the formfactor program, *i.e.* the reconstruction of quantum fields from a scattering matrix [3, 26, 38]; for more recent progress on this issue see also [1, 2, 17].

The second step in this approach consists in showing that, besides field operators localized in wedges, there appear also local observables, *i.e.* operators which are localized in compact spacetime regions, such as double cones. As any double cone in two dimensions is the intersection of two opposite wedges, local observables ought to be elements of the intersection of wedge algebras. The question of whether these intersections are non–trivial turned out to be a difficult one, however, and has not yet been settled. Some ideas as to how this problem may be tackled in models are discussed in [37].

It is the aim of the present letter to point out an alternative strategy for the proof of the non–triviality of the intersections of wedge algebras. By combining results scattered in the literature and casting them into a simple condition, we will show that the non–triviality of these intersections can be deduced from spectral (nuclearity) properties of the modular operators on certain specific subspaces of the Hilbert space. Thus the algebraic problem of determining intersections of wedge algebras amounts to a problem in spectral analysis which seems to be better tractable.

The subsequent section contains an abstract version of our nuclearity condition and a discussion of its consequences in a general algebraic setting. In Section 3 these results are carried over to a family of theories with factorizing S–matrix in two–dimensional Minkowski space. It is shown that compactly localized operators exist in any theory complying with our condition. Section 4 illustrates the type of computations needed to verify this condition in a simple example. The article closes with a brief outlook.

# 2 Modular nuclearity and its consequences

In this section we present our nuclearity condition in a general setting, extracted from the more concrete structures in field theoretic models, and discuss its implications. We begin by introducing our notation and listing our assumptions.

(a) Let $\mathcal{H}$ be a Hilbert space and let $U$ be a continuous unitary representation of $\mathbb{R}^2$ acting on $\mathcal{H}$. Choosing proper coordinates on $\mathbb{R}^2$, $x = (x_0, x_1)$, the joint spectrum of the corresponding generators $(P_0, P_1)$ of $U$ is contained in the cone $V_+ \doteq \{(p_0, p_1) \in \mathbb{R}^2 : p_0 \geq |p_1|\}$ and there is an (up to a phase unique) unit vector $\Omega \in \mathcal{H}$ which is invariant under the action of $U$.

(b) There is a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ such that for each element $x$ of the wedge $W \doteq \{y \in \mathbb{R}^2 : |y_0| + y_1 < 0\}$ the adjoint action of the unitaries $U(x)$ induces endomorphisms of $\mathcal{M}$,

$$\mathcal{M}(x) \doteq U(x)\mathcal{M}U(x)^{-1} \subset \mathcal{M}, \quad x \in W. \tag{2.1}$$

Moreover, $\Omega$ is cyclic and separating for $\mathcal{M}$.

It is well known that, under these circumstances, the algebraic properties of $\mathcal{M}$ are strongly restricted. As a matter of fact, disregarding the trivial possibility that $\mathcal{H}$ is one–dimensional and $\mathcal{M} = \mathbb{C}$, the following result has been established in [28, Thm. 3].

**Lemma 2.1.** *Under the preceding two conditions the algebra $\mathcal{M}$ is a factor of type $III_1$ according to the classification of Connes.*

It immediately follows from this result that the algebras $\mathcal{M}(x)$ are factors of type $\mathrm{III}_1$ as well. Little is known, however, about the algebraic structure of the relative commutants $\mathcal{M}(x)' \cap \mathcal{M}$ of $\mathcal{M}(x)$ in $\mathcal{M}$, $x \in W$. Even the question of whether these relative commutants are non–trivial has not been settled in this general setting. Yet this question turns out to have an affirmative answer and, as a matter of fact, the algebraic structures are completely fixed if the inclusions (2.1) are split, *i.e.* if for each $x \in W$ there is a factor $\mathcal{N}$ of type $\mathrm{I}_\infty$ such that

$$\mathcal{M}(x) \subset \mathcal{N} \subset \mathcal{M}. \tag{2.2}$$

First, the split property implies that $\mathcal{M}$ is isomorphic to the unique [24] hyperfinite factor of type $\mathrm{III}_1$. We briefly recall here the argument: As $\Omega$ is cyclic and separating for $\mathcal{M}$, and hence for $\mathcal{M}(x)$, this is also true for $\mathcal{N}$. It follows that $\mathcal{N}$, being of type $\mathrm{I}_\infty$, is separable in the ultraweak topology and consequently $\mathcal{H}$ is separable, cf. [21, Prop. 1.2]. Now, as $U$ is continuous, $\mathcal{M}$ is continuous from the inside, $\mathcal{M} = \bigvee_{x\in W} \mathcal{M}(x)$. The split property thus implies that $\mathcal{M}$ can be approximated from the inside by separable type $\mathrm{I}_\infty$ factors and therefore is hyperfinite, cf. [11, Prop. 3.1]. Knowing also that it is of type $\mathrm{III}_1$, the assertion follows.

Secondly, the split property implies that $\mathcal{M}(x)' \cap \mathcal{M}$, $x \in W$, is isomorphic to the hyperfinite factor of type $\mathrm{III}_1$ as well. This can be seen as follows [21]. On a separable Hilbert space $\mathcal{H}$, any factor of type III has cyclic and separating vectors [34, Cor. 2.9.28]. Moreover, for *any* von Neumann algebra on $\mathcal{H}$ with a cyclic and a separating vector there exists a dense $G_\delta$ set of vectors which are both, cyclic and separating [20]. Now, taking into account that $\mathcal{N}$ is isomorphic to $\mathcal{B}(\mathcal{H})$, the relative commutant $\mathcal{M}(x)' \cap \mathcal{N}$ of the type III factor $\mathcal{M}(x)$ in $\mathcal{N}$ is (anti)isomorphic to $\mathcal{M}(x)$ by Tomita–Takesaki theory. It is therefore of type III and has cyclic vectors in $\mathcal{H}$. This holds *a fortiori* for $\mathcal{M}(x)' \cap \mathcal{M} \supset \mathcal{M}(x)' \cap \mathcal{N}$ and, as $\Omega$ is separating for $\mathcal{M}$, the relative commutant $\mathcal{M}(x)' \cap \mathcal{M}$ has a dense $G_\delta$ set of cyclic and separating vectors. But the intersection of a finite number of dense $G_\delta$ sets is non–empty. So we conclude that the triple $\mathcal{M}$, $\mathcal{M}(x)$ and $\mathcal{M}(x)' \cap \mathcal{M}$ has a joint cyclic and separating vector in $\mathcal{H}$. The inclusion (2.2) is thus a standard split inclusion according to the terminology in [21]. In particular, there is a spatial isomorphism mapping $\mathcal{M}(x) \bigvee \mathcal{M}'$ on $\mathcal{H}$ onto $\mathcal{M}(x) \otimes \mathcal{M}'$ on $\mathcal{H} \otimes \mathcal{H}$ [19]. By taking commutants, we conclude that $\mathcal{M}(x)' \cap \mathcal{M}$ is isomorphic to $\mathcal{M}(x)' \otimes \mathcal{M}$, $x \in W$. The statement about the algebraic structure of the relative commutant then follows.

It seems difficult, however, to establish the existence of intermediate type $\mathrm{I}_\infty$ factors $\mathcal{N}$ in the inclusions (2.2) for concretely given $\{\mathcal{M}, U, \mathcal{H}\}$, and this may be the reason why this strategy of establishing the non–triviality of relative commutants has been discarded in [37]. Yet the situation is actually not hopeless, the interesting point being that the existence of the desired factors can be derived from spectral properties of the modular operator $\Delta$ affiliated with the pair $(\mathcal{M}, \Omega)$. Recalling that a linear map from a Banach space into another one is said to be nuclear if it can be decomposed into a series of maps of rank one whose norms are summable, we extract the following pertinent condition from [12].

(c) Modular Nuclearity Condition: For any given $x \in W$ the map

$$M \mapsto \Delta^{1/4} M\Omega, \quad M \in \mathcal{M}(x), \tag{2.3}$$

is nuclear. Equivalently, since $\Delta^{1/4}$ is invertible, the image of the unit ball in $\mathcal{M}(x)$ under this map is a nuclear subset of $\mathcal{H}$.

Since $\Omega$ is cyclic and separating for $\mathcal{M}$ and the algebras $\mathcal{M}(x)$, $\mathcal{M}$ both are factors, it follows from the modular nuclearity condition (c) that the inclusions $\mathcal{M}(x) \subset \mathcal{M}$, $x \in W$, are split [12, Thm. 3.3]. Conversely, if these inclusions are split, the map (2.3) has to be compact, at least. Thus a proof of the split property (2.2) amounts to a spectral analysis of the operator $\Delta^{1/4}$ on the subspaces $\mathcal{M}(x)\,\Omega \subset \mathcal{H}$. This task is, as we shall see, manageable in concrete applications. We summarize the results of the preceding discussion in the following proposition.

**Proposition 2.2.** *Let $\{\mathcal{M}, U, \mathcal{H}\}$ be a triple satisfying conditions (a), (b) and (c), stated above. Then, for $x \in W$,*

*(i) the inclusion $\mathcal{M}(x) \subset \mathcal{M}$ is split;*

*(ii) the relative commutant $\mathcal{M}(x)' \cap \mathcal{M}$ is isomorphic to the unique hyperfinite type $III_1$ factor. In particular, it has cyclic and separating vectors.*

We conclude this section by noting that any triple $\{\mathcal{M}, U, \mathcal{H}\}$ as in the preceding proposition can be used to construct a non–trivial Poincaré covariant net of local algebras on two–dimensional Minkowski space $\mathbb{R}^2$. Following closely the discussion in [5, 6], we first note that the modular group $\Delta^{is}, s \in \mathbb{R}$, and the modular conjugation $J$ affiliated with $(\mathcal{M}, \Omega)$ can be interpreted as representations of proper Lorentz transformations $\Lambda$ (having determinant one). More specifically, if $\Lambda$ is any such transformation and $\Lambda = (-1)^\sigma B(\theta)$ its polar decomposition, where $\sigma \in \{0, 1\}$ and $B(\theta)$ is a boost with rapidity $\theta \in \mathbb{R}$, one can show that

$$U(x, \Lambda) \doteq U(x)\, J^\sigma\, \Delta^{i\theta/2\pi} \tag{2.4}$$

defines a continuous (anti)unitary representation of the proper Poincaré group [5]. Moreover, $\Omega$ is invariant under the action of these operators and may thus be interpreted as a vacuum state. Setting $\mathcal{R}(\Lambda W + x) \doteq U(x, \Lambda)\mathcal{M}U(x, \Lambda)^{-1}$, one obtains a local (as a matter of fact, Haag–dual) Poincaré covariant net of wedge algebras on $\mathbb{R}^2$. Denoting the double cones in $\mathbb{R}^2$ by $C_{x,y} \doteq (-W+x) \cap (W+y)$, $x-y \in W$, the corresponding algebras

$$\mathcal{R}(C_{x,y}) \doteq \mathcal{R}(W + x)' \cap \mathcal{R}(-W + y)' = \mathcal{M}(x)' \cap \mathcal{M}(y) \tag{2.5}$$

are non–trivial according to the preceding proposition. As was shown in [5], they form a local net on $\mathbb{R}^2$ which is relatively local to the wedge algebras and transforms covariantly under the adjoint action of $U(x, \Lambda)$. It may thus be interpreted as a net of local observables. The vacuum vector $\Omega$ need not be cyclic for the local algebras, however. In fact, thinking of theories exhibiting solitonic excitations of $\Omega$ which are localized in wedge regions, this may also not be expected in general.

# 3 Applications to field theoretic models

We carry over now the results of the preceding section to the framework of two–dimensional models and indicate their significance for the formfactor program, *i.e.* the reconstruction of local observables and fields from a given factorizing scattering matrix.

For the sake of concreteness, we restrict attention here to the theory of a single massive particle with given two–particle scattering function $S_2$, as considered in [35,36] and described in more detail in [27]. The Hilbert space of the theory is conveniently represented as the $S_2$–symmetrized Fock space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$. Here the subspace $\mathcal{H}_0$ consists of multiples of the vacuum vector $\Omega$ and, using the parameterization of the mass shell by the rapidity $\theta$,

$$p(\theta) = m\,\big(\mathrm{ch}(\theta), \mathrm{sh}(\theta)\big), \quad \theta \in \mathbb{R}, \tag{3.1}$$

the single particle space $\mathcal{H}_1$ can be identified with the space of square integrable functions $\theta \mapsto \Psi_1(\theta)$ with norm given by

$$\|\Psi_1\|^2 = \int d\theta\, |\Psi_1(\theta)|^2. \tag{3.2}$$

The elements of the $n$–particle space $\mathcal{H}_n$ are represented by square integrable functions $\theta_1 \dots \theta_n \mapsto \Psi_n(\theta_1, \dots, \theta_n)$ which are $S_2$–symmetric,

$$\Psi_n(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) = S_2(\theta_i - \theta_{i+1})\, \Psi_n(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n). \tag{3.3}$$

Here $\zeta \mapsto S_2(\zeta)$ is the scattering function which is continuous and bounded on the strip $\{\zeta \in \mathbb{C} : 0 \le \mathrm{Im}\,\zeta \le \pi\}$, analytic in its interior and satisfies, for $\theta \in \mathbb{R}$, the unitarity and crossing relations

$$S_2(\theta)^{-1} = \overline{S_2(\theta)} = S_2(-\theta) = S_2(\theta + i\pi). \tag{3.4}$$

On $\mathcal{H}$ there acts a continuous unitary representation $U$ of the proper orthochronous Poincaré group, given by

$$\big(U(x, B(\theta))\, \Psi\big)_n(\theta_1, \dots, \theta_n) \doteq e^{ix \sum_{j=1}^n p(\theta_j)}\ \Psi_n(\theta_1 - \theta, \dots, \theta_n - \theta). \tag{3.5}$$

It satisfies the relativistic spectrum condition, *i.e.* the joint spectrum of the generators $P = (P_0, P_1)$ of the translations $U(\mathbb{R}^2, 1)$ is contained in $V_+$. Moreover, there is an antiunitary operator $J$ on $\mathcal{H}$, representing the PCT symmetry. It is given by

$$\big(J\,\Psi\big)_n(\theta_1, \dots, \theta_n) \doteq \overline{\Psi_n(\theta_n, \dots, \theta_1)}. \tag{3.6}$$

As in the case of the bosonic and fermionic Fock spaces, one can define creation and annihilation operators $z^\dagger(\theta)$, $z(\theta)$ (in the sense of operator valued distributions) on the dense subspace $\mathcal{D} \subset \mathcal{H}$ of vectors with a finite particle number. They are hermitian conjugates with respect to each other and satisfy the Fadeev–Zamolodchikov relations

$$z^\dagger(\theta) z^\dagger(\theta') = S_2(\theta - \theta')\, z^\dagger(\theta') z^\dagger(\theta), \quad z(\theta) z(\theta') = S_2(\theta - \theta')\, z(\theta') z(\theta),$$
$$z(\theta) z^\dagger(\theta') = S_2(\theta' - \theta)\, z^\dagger(\theta') z(\theta) + \delta(\theta - \theta')\, 1. \tag{3.7}$$

Their action on $\mathcal{D}$ is fixed by the equations

$$(z^\dagger(\theta_1)\dots z^\dagger(\theta_n)\,\Omega,\Psi) = (n!)^{1/2}\,\Psi_n(\theta_1,\dots,\theta_n), \quad z(\theta)\,\Omega = 0. \tag{3.8}$$

With the help of these creation and annihilation operators one can define on $\mathcal{D}$ a field $\phi$, setting

$$\phi(f) \doteq z^\dagger(f_+) + z(f_-), \quad f \in \mathcal{S}(\mathbb{R}^2), \tag{3.9}$$

where

$$f_\pm(\theta) \doteq (2\pi)^{-1}\int dx\, f(x)\, e^{\pm ip(\theta)x} \tag{3.10}$$

and we adopt the convention that, both, $z^\dagger(\,\cdot\,)$ and $z(\,\cdot\,)$ are complex linear on the space of test functions.

It has been shown in [27] that $\phi$ transforms covariantly under the adjoint action of the proper orthochronous Poincaré group,

$$U(x,B)\,\phi(f)\,U(x,B)^{-1} = \phi(f_{x,B}), \tag{3.11}$$

where $f_{x,B}(y) \doteq f(B^{-1}(y-x))$, $y \in \mathbb{R}^2$. Moreover, $\phi$ is real, $\phi(f)^* \supset \phi(\overline{f})$, and each vector in $\mathcal{D}$ is entire analytic for the operators $\phi(f)$. Since $\mathcal{D}$ is stable under their action, these operators are essentially selfadjoint on this domain for real $f$. We mention as an aside that the fields $\phi(f)$ are polarization–free generators in the sense of [7].

Denoting the selfadjoint extensions of $\phi(f)$, $f$ real, by the same symbol, one can define the von Neumann algebras

$$\mathcal{R}(W+x) \doteq \{e^{i\phi(f)} : \mathrm{supp} f \subset W+x\}'', \quad x \in \mathbb{R}^2, \tag{3.12}$$

where $W$ denotes, as before, the wedge $W \doteq \{y \in \mathbb{R}^2 : |y_0| + y_1 < 0\}$. With the help of the PCT operator $J$ one can also define algebras corresponding to the opposite wedges,

$$\mathcal{R}(-W-x) \doteq J\,\mathcal{R}(W+x)\,J, \quad x \in \mathbb{R}^2. \tag{3.13}$$

Now, given an arbitrary proper Lorentz transformation $\Lambda$ with polar decomposition $\Lambda = (-1)^\sigma B$, $\sigma \in \{0,1\}$, one obtains a representation of the proper Poincaré group, setting $U(x,\Lambda) \doteq U(x,B)J^\sigma$. It then follows from the covariance properties (3.11) of the field that

$$U(x,\Lambda)\,\mathcal{R}(\pm W + y)U(x,\Lambda)^{-1} = \mathcal{R}(\pm(-1)^\sigma W + \Lambda y + x), \tag{3.14}$$

taking into account that the wedge $W$ is stable under the action of boosts. So, by this construction, one arrives at a Poincaré covariant net of wedge algebras on two–dimensional Minkowski space.

It has been shown in [27] that this net is local,

$$\mathcal{R}(\pm W + x) \subset \mathcal{R}(\mp W + x)', \tag{3.15}$$

and that $\Omega$ is cyclic and separating for the wedge algebras (and hence for their commutants).

The triple $\{\mathcal{R}(W), U(\mathbb{R}^2,1), \mathcal{H}\}$ satisfies conditions (a) and (b) given in the preceding section. More can be said by making use of modular theory and certain specific domain properties of the field $\phi$.

**Proposition 3.1.** *Let $\mathcal{R}(W)$ be the algebra defined above. Then*
*(i) the modular group and conjugation affiliated with $(\mathcal{R}(W), \Omega)$ are given by $\mathbb{R} \ni \lambda \mapsto U(0, B(2\pi\lambda))$ and $J$, respectively;*
*(ii) $\mathcal{R}(W)' = \mathcal{R}(-W)$ (Haag duality).*

*Proof.* Let $\Delta_W$, $J_W$ be the modular operator and conjugation, respectively, affiliated with $(\mathcal{R}(W), \Omega)$. It follows from modular theory that any boost $U(0, B)$ commutes with $\Delta_W$ and $J_W$ since $\Omega$ is invariant and $\mathcal{R}(W)$ is stable under its (adjoint) action. Hence $\lambda \mapsto V(\lambda) \doteq U(0, B(2\pi\lambda))\Delta_W^{-i\lambda}$ is a continuous unitary representation of $\mathbb{R}$ with the latter properties. Moreover, $V(\lambda)$ commutes with all boosts $U(0, B)$ and, by a theorem of Borchers [5], also with all translations $U(x, 1)$. Since the restriction of $U$ to the proper orthochronous Poincaré group acts irreducibly on $\mathcal{H}_1$, one concludes that $V(\lambda) \upharpoonright \mathcal{H}_1 = e^{i\lambda c} 1$ for fixed real $c$ and any $\lambda \in \mathbb{R}$.

Now, for real $f$ with supp$f \subset W$, $\phi(f)$ is a selfadjoint operator affiliated with $\mathcal{R}(W)$, and the same holds for $\phi_\lambda(f) \doteq V(\lambda)\phi(f)V(\lambda)^{-1}$, $\lambda \in \mathbb{R}$, because of the stability of $\mathcal{R}(W)$ under the adjoint action of $V(\lambda)$. So both operators commute with all elements of $\mathcal{R}(W)'$. Since $\Omega$ is invariant under the action of $V(\lambda)^{-1}$ and since $\phi(f)\Omega \in \mathcal{H}_1$, the preceding result implies

$$\big(\phi_\lambda(f) - e^{i\lambda c}\phi(f)\big)A'\Omega = 0, \quad A' \in \mathcal{R}(W)'. \tag{3.16}$$

It will be shown below that the dense set of vectors $\mathcal{R}(W)'\Omega$ is a core, both, for $\phi(f)$ and $\phi_\lambda(f)$. Hence $\phi_\lambda(f) = e^{i\lambda c}\phi(f)$ which, in view of the selfadjointness of the field operators, is only possible if $c = 0$. This holds for any choice of $f$ within the above limitations, so $V(\lambda)$ acts trivially on $\mathcal{R}(W)$. Taking also into account that $\Omega$ is cyclic for $\mathcal{R}(W)$, one arrives at $V(\lambda) = 1$, $\lambda \in \mathbb{R}$, from which the first part of statement $(i)$ follows.

Similarly, modular theory and the theorem of Borchers mentioned above imply that the unitary operator $I \doteq J_W J$ commutes with all Poincaré transformations $U(x, B)$ and, taking into account relation (3.15), one also has $I\mathcal{R}(W)I^{-1} \subset \mathcal{R}(W)$. Hence, putting $\phi_I(f) \doteq I\phi(f)I^{-1}$, one finds by the same reasoning as in the preceding step that $\phi_I(f) = \phi(f)$. Thus $I = 1$, proving the second part of statement $(i)$. The statement about Haag duality then follows from the equalities

$$\mathcal{R}(W)' = J_W\mathcal{R}(W)J_W = J\mathcal{R}(W)J = \mathcal{R}(-W). \tag{3.17}$$

It remains to prove the assertion that $\mathcal{R}(W)'\Omega$ is a core for the selfadjoint operators $\phi(f)$, $\phi_\lambda(f)$ and $\phi_I(f)$, respectively. To this end one makes use of bounds, given in [27], on the action of the field operators on $n$–particle states $\Psi_n$. One has $\|\phi(f)\Psi_n\| \leq c_f(n+1)^{1/2}\|\Psi_n\|$, where $c_f$ is some constant depending only on $f$. Since the field operators change the particle number at most by $\pm 1$, one can proceed from this estimate to corresponding bounds for $\Psi \in \mathcal{D}$, given by $\|\phi(f)\Psi\| \leq 2c_f\, \|(N + 1)^{1/2}\Psi\|$, where $N$ is the particle number operator. Recalling that $P_0$ denotes the (positive) generator of the time translations, it is also clear that $m\,(N + 1) \leq (P_0 + m1)$. So for $\Psi \in \mathcal{D} \cap \mathcal{D}_0$, where $\mathcal{D}_0$ is the domain of $P_0$, one arrives at the inequalities

$$\|\phi(f)\Psi\| \leq 2c_f\, \|(N + 1)^{1/2}\Psi\| \leq 2m^{-1/2}c_f\, \|(P_0 + m1)^{1/2}\Psi\|. \tag{3.18}$$

It follows from this estimate by standard arguments that any core for $P_0$ is also a core for the field operators $\phi(f)$. Since the unitary operators $V(\lambda)$ and $I$ in the preceding steps were shown to commute with the time translations, this domain property is also shared by the transformed field operators $\phi_\lambda(f)$ and $\phi_I(f)$, respectively.

In order to complete the proof, one has only to show that $\mathcal{R}(W)'\Omega \cap \mathcal{D}_0$ is a core for $P_0$. Now $\mathcal{R}(W)'\Omega$ is mapped into itself by all translations $U(x)$, $x \in -W$. Hence, taking into account the invariance of $\Omega$ under translations, one finds that $\widetilde{f}(P)\mathcal{R}(W)'\Omega \subset \mathcal{R}(W)'\Omega \cap \mathcal{D}_0$ for any test function $f$ with supp$f \subset -W$. But this space of functions contains elements $f$ such that $\widetilde{f}(P)$ is invertible. Hence $(P_0 \pm i1)\widetilde{f}(P)\mathcal{R}(W)'\Omega \subset (P_0 \pm i1)(\mathcal{R}(W)'\Omega \cap \mathcal{D}_0)$ both are dense subspaces of $\mathcal{H}$, proving the statement. □

In view of the covariance properties of the net, it is apparent that analogous statements hold for all wedge algebras. Thus the only point left open in this reconstruction of a relativistic quantum field theory from scattering data is the question of whether the wedge algebras contain operators which can be interpreted as observables localized in finite spacetime regions, such as the double cones $C_{x,y} \doteq (W+y) \cap (-W+x)$, $x-y \in W$. By Einstein causality, observables localized in $C_{x,y}$ have to commute with all operators localized in the adjacent wedges $W+x$ and $-W+y$. They are therefore elements of the algebra

$$\mathcal{R}(C_{x,y}) \doteq \mathcal{R}(W+x)' \cap \mathcal{R}(-W+y)' = \mathcal{R}(-W+x) \cap \mathcal{R}(W+y). \quad (3.19)$$

It follows from the properties of the wedge algebras established thus far that the resulting map $\mathcal{C} \mapsto \mathcal{R}(C)$ from double cones to von Neumann algebras defines a local and Poincaré covariant net on Minkowski space. So if the theory describes local observables, the algebras $\mathcal{R}(C)$ are to be non–trivial.

At this point the nuclearity condition formulated in Sec. 2 comes in. Knowing by the preceding proposition the explicit form of the modular operator affiliated with $(\mathcal{R}(W), \Omega)$ and taking into account the invariance of $\Omega$ under spacetime translations, we are led to consider, for given $x \in W$, the maps

$$A \mapsto U(0, B(-i\pi/2))\, U(x,1)\, A\Omega, \quad A \in \mathcal{R}(W). \quad (3.20)$$

Within the present context one then has the following more concrete version of Proposition 2.2.

**Proposition 3.2.** *Let the maps (3.20) be nuclear, $x \in W$. Then*

*(i) the net of wedge algebras has the split property;*

*(ii) for any open double cone $C \subset \mathbb{R}^2$ the corresponding algebra $\mathcal{R}(C)$ is isomorphic to the unique hyperfinite factor of type $III_1$. In particular it has cyclic and separating vectors.*

So in order to establish the existence of local operators in the theory, one needs an estimate of the size of the set of vectors

$$U(0, B(-i\pi/2))\, U(x,1)\, A\Omega, \quad A \in \mathcal{R}(W)_1, \quad (3.21)$$

*i.e.* the image of the unit ball $\mathcal{R}(W)_1$ under the action of the map (3.20). We briefly indicate here the steps required in such an analysis which are similar to those carried out in [8] in an investigation of the Haag–Swieca compactness condition; a more detailed account of these results will be presented elsewhere.

Making use of the localization properties of the operators $A \in \mathcal{R}(W)$ and the analyticity properties of the scattering function $S_2$, one can show that the $n$–particle wave functions

$$\theta_1, \dots \theta_n \mapsto (A\Omega)_n(\theta_1, \dots \theta_n) \tag{3.22}$$

extend, in the sense of distributions, to analytic functions in the domain $0 < \mathrm{Im}\,\theta_i < \pi$, $-\delta < \mathrm{Im}\,(\theta_i - \theta_k) < \delta$, where $i, k = 1, \dots n$ and $\delta$ depends on the domain of analyticity of the scattering function $S_2$. Thus the functions

$$\begin{aligned}\theta_1, \dots \theta_n \mapsto & \,(U(0, B(-i\pi/2))\, A\Omega)_n(\theta_1, \dots \theta_n) \\ & = (A\Omega)_n(\theta_1 + i\pi/2, \dots \theta_n + i\pi/2)\end{aligned} \tag{3.23}$$

are analytic in the domain $-\delta/n < \mathrm{Im}\,\theta_i < \delta/n,\ \ i = 1, \dots n$. As a matter of fact, if $A \in \mathcal{R}(W)_1$, the family of these functions turns out to be uniformly bounded (normal) on this domain. Taking also into account that $U$ is a representation of the Poincaré group, one obtains for $x \in W$ the equality

$$U(0, B(-i\pi/2))\, U(x, 1)\, A\Omega = e^{x_1 P_0 - x_0 P_1}\, U(0, B(-i\pi/2))\, A\Omega, \tag{3.24}$$

so the $n$–particle components of the vectors (3.21) have wave functions of the form

$$\begin{aligned}\theta_1, \dots \theta_n \mapsto & \,(U(0, B(-i\pi/2))\, U(x, 1)\, A\Omega)_n(\theta_1, \dots \theta_n) \\ & = e^{m \sum_{k=1}^n (x_1 \,\mathrm{ch}(\theta_k) - x_0 \,\mathrm{sh}(\theta_k))}\, (A\Omega)_n(\theta_1 + i\pi/2, \dots \theta_n + i\pi/2).\end{aligned} \tag{3.25}$$

Since, for $x \in W$, the exponential factor gives rise to a strong damping of large rapidities, it follows from the preceding results that the wave functions (3.25) form, for $A \in \mathcal{R}(W)_1$, a bounded subset of the space of test functions $\mathcal{S}(\mathbb{R}^n)$ and hence a nuclear subset of $L^2(\mathbb{R}^n) = \mathcal{H}_n$. Moreover, taking into account the spectral properties of $(P_0, P_1)$, relation (3.24) combined with the estimate $\|U(0, B(-i\pi/2))\, A\Omega\| \leq \|A\|$ following from modular theory implies

$$\|(U(0, B(-i\pi/2))\, U(x, 1)\, A\Omega)_n\| \leq e^{n\, m(x_1 + |x_0|)}, \quad A \in \mathcal{R}(W)_1. \tag{3.26}$$

So these norms tend rapidly to 0 for large $n \in \mathbb{N}$ if $x \in W$. Combining these facts, one finds after a moments reflection that the sets (3.21) are relatively compact in $\mathcal{H}$, implying that the maps (3.20) are compact. So they can be approximated with arbitrary precision by finite sums of maps of rank one. In order to prove that they are also nuclear, one needs more refined estimates, however.

# 4 An instructive example

In order to illustrate the quantitative estimates needed for the proof that the map (3.20) is nuclear, we consider here the case of trivial scattering, $S_2 = 1$, *i.e.* the theory of a free massive Bose field $\phi$. There the combinatorial problems appearing in the analysis of the size of sets of the type (3.21) have been settled in [15] and we shall make use of these results here.

We begin by recalling some well known facts: The restrictions of the field $\phi$ and of its time derivative $\dot{\phi}$ to the time zero plane are operator valued distributions on the domain $\mathcal{D}$. These time zero fields, commonly denoted by $\varphi$ and $\pi$, satisfy canonical equal time commutation relations. If smeared with test functions $h$ having support in the interval $(-\infty, 0)$, they generate the von Neumann algebra $\mathcal{R}(W)$ and, applying them to the vacuum vector $\Omega$, they create closed subspaces $\mathcal{L}_\varphi(W)$, $\mathcal{L}_\pi(W)$ of the single particle space $\mathcal{H}_1$ given by

$$\begin{aligned} \mathcal{L}_\varphi(W) &= \{\theta \mapsto \widetilde{h}(m\,\mathrm{sh}(\theta)) : \mathrm{supp}\, h \subset (-\infty, 0)\}^-, \\ \mathcal{L}_\pi(W) &= \{\theta \mapsto \mathrm{ch}(\theta)\, \widetilde{h}(m\,\mathrm{sh}(\theta)) : \mathrm{supp}\, h \subset (-\infty, 0)\}^-, \end{aligned} \tag{4.1}$$

where the tilde denotes Fourier transformation. We also consider the shifted subspaces $\mathcal{L}_\varphi(W + x) \doteq U(x,1)\, \mathcal{L}_\varphi(W)$ and $\mathcal{L}_\pi(W + x) \doteq U(x,1)\, \mathcal{L}_\pi(W)$ and denote the corresponding orthogonal projections by $E_\varphi(W + x)$ and $E_\pi(W + x)$, respectively. After these preparations we are in a position to apply the results in [15, Thm. 2.1] which we recall here for the convenience of the reader in a form appropriate for the present investigation.

**Lemma 4.1.** *Consider the theory with scattering function $S_2 = 1$ and let $E_\varphi(W + x)\, U(0, B(-i\pi/2))$ and $E_\pi(W + x)\, U(0, B(-i\pi/2))$ be trace class operators with operator norms less than 1, $x \in W$. Then the sets (3.21) are nuclear.*

Thus the proof that the modular nuclearity condition is satisfied in the present theory reduces to a problem of spectral analysis in the single particle space $\mathcal{H}_1$. We first turn to the task of providing estimates of the norms of the operators appearing in the lemma.

Let $\Phi_h \in \mathcal{L}_\varphi(W)$ be a vector with wave function $\theta \mapsto \Phi_h(\theta) = \widetilde{h}(m\,\mathrm{sh}(\theta))$, where $h$ is, as before, a test function with support in $(-\infty, 0)$. Because of these support properties, $\Phi_h$ lies in the domain of all boosts $U(0, B(\theta))$ for complex $\theta$ with $-\pi \leq \mathrm{Im}\,\theta \leq 0$. Furthermore, as $\mathrm{sh}(\theta + i\pi) = \mathrm{sh}(-\theta)$, one has $(U(0, B(-i\pi))\Phi_h)(\theta) = \widetilde{h}(m\,\mathrm{sh}(-\theta)) = \Phi_h(-\theta)$. But this implies $\|U(0, B(-i\pi))\, \Phi_h\| = \|\Phi_h\|$ and consequently $\|U(0, B(-i\pi/2))\, \Phi_h\| \leq \|\Phi_h\|$. Making use now of the properties of the representation $U$, one obtains the estimate, $x \in W$,

$$\begin{aligned} &\|U(0, B(-i\pi/2))\, U(x,1)\, \Phi_h\| = \|e^{x_1 P_0 - x_0 P_1}\, U(0, B(-i\pi/2))\, \Phi_h\| \\ &\leq e^{m(x_1 + |x_0|)}\, \|U(0, B(-i\pi/2))\, \Phi_h\| \leq e^{m(x_1 + |x_0|)}\, \|U(x,1)\, \Phi_h\|. \end{aligned} \tag{4.2}$$

Since $\Phi_h$ was arbitrary within the above limitations and $(x_1 + |x_0|)$ is negative, this yields the norm estimate $\|U(0, B(-i\pi/2))\, E_\varphi(W + x)\| < 1$, $x \in W$.

But the adjoint operator $E_\varphi(W+x)\,U(0,B(-i\pi/2))$ has the same norm, so the desired bound follows. In a similar manner one can show that the operator $E_\pi(W+x)\,U(0,B(-i\pi/2))$ also has norm less than 1.

It remains to establish the trace class property of these operators. To this end we consider the restriction of the operator $U(0,B(-i\pi/2))\,U(x,1)$, $x \in W$, to the subspaces $\mathcal{L}_\varphi(W)$ and $\mathcal{L}_\pi(W)$, respectively. Let, as before, $\Phi_h \in \mathcal{L}_\varphi(W)$, then

$$(U(0,B(-i\pi/2))\,U(x,1)\,\Phi_h)(\theta) = e^{x_1 p_0(\theta)\,-\,x_0 p_1(\theta)}\,\Phi_h(\theta+i\pi/2). \quad (4.3)$$

Making use of the analyticity and boundedness properties of $\theta \mapsto \Phi_h(\theta)$ and the fact that $\Phi_h(\theta+i\pi) = \Phi_h(-\theta)$, one can represent $\Phi_h(\theta+i\pi/2)$ by a Cauchy integral,

$$\Phi_h(\theta+i\pi/2) = \frac{1}{2\pi i}\int d\theta' \left\{\frac{1}{\theta'-\theta-i\pi/2} + \frac{1}{\theta'+\theta-i\pi/2}\right\}\Phi_h(\theta'). \quad (4.4)$$

Next, for $x \in W$, let $X_\varphi$ be the operator on $\mathcal{H}_1$ with kernel

$$X_\varphi(\theta,\theta') = \frac{1}{2\pi i}\,e^{x_1 p_0(\theta)\,-\,x_0 p_1(\theta)}\left\{\frac{1}{\theta'-\theta-i\pi/2} + \frac{1}{\theta'+\theta-i\pi/2}\right\}. \quad (4.5)$$

Being the sum of products of multiplication operators in rapidity space, respectively its dual space, which are bounded and rapidly decreasing, it is apparent that $X_\varphi$ is of trace class. Moreover, by the preceding results, $U(0,B(-i\pi/2))\,E_\varphi(W+x) = X_\varphi\,E_\varphi(W)\,U(x,1)^{-1}$. Since the trace class operators form a *–ideal in $\mathcal{B}(\mathcal{H}_1)$, if follows that $U(0,B(-i\pi/2))\,E_\varphi(W+x)$ and its adjoint $E_\varphi(W+x)\,U(0,B(-i\pi/2))$ are of trace class.

By a similar argument one can also establish the trace class property of $E_\pi(W+x)\,U(0,B(-i\pi/2))$, the only difference being that for vectors $\Phi_h \in \mathcal{L}_\pi(W)$ with wave functions $\theta \mapsto \Phi_h(\theta) = \mathrm{ch}(\theta)\,\widetilde{h}(m\,\mathrm{sh}(\theta))$ one now has $\Phi_h(\theta+i\pi) = -\Phi_h(-\theta)$. As a consequence, the sum in relation (4.4) turns into a difference, but this does not affect the conclusions. So the following statement has been proven.

**Proposition 4.2.** *In the theory with scattering function $S_2 = 1$, the sets (3.21) and corresponding maps*

$$A \mapsto U(0,B(-i\pi/2))\,U(x,1)\,A\Omega, \quad A \in \mathcal{R}(W). \quad (4.6)$$

*are nuclear, $x \in W$.*

We thus have verified in the present model the modular nuclearity condition for wedge algebras with all of its consequences. In particular, the wedge algebras have the split property. Although the latter fact was known before [29], there did not yet exist a proof in the literature.

By similar arguments one can also treat the theory with scattering function $S_2 = -1$, *i.e.* the theory of a (non–local) free massive Fermi field. There one expects that the sets (3.21) are somewhat smaller than in the present case because of the Pauli principle. It is even more challenging, however, to provide a quantitative estimate of the size of the sets (3.21) in theories with generic scattering function. This problem will be tackled elsewhere.

## 5 Conclusions

Within the algebraic setting of quantum field theory, we have presented a method which allows one to decide whether algebras affiliated with wedge shaped regions in two–dimensional Minkowski space contain compactly localized operators. This method seems to be particularly useful for proving the existence of local operators in theories with factorizing $S$–matrix. It is thus complementary to the formfactor program, where one tries to exhibit such operators explicitly by solving an infinite system of equations.

The upshot of the present investigation is the insight that the basic algebraic problem of checking locality, which amounts to computing relative commutants, can be replaced by an analysis of spectral properties of representations of the Poincaré group. There exist other methods by which the crucial intermediate step in our argument, the proof of the split property of wedge algebras, could be accomplished [10,13,16,18,19,30]. But the present approach requires less *a priori* information about the underlying theory and also seems better managable in concrete applications. Moreover, in view of the fact that it relies only on the modular structure, it is applicable to theories on arbitrary spacetime manifolds.

It is apparent, however, that the split property of wedge algebras is in general an unnecessarily strong requirement if one is merely interested in the existence of compactly localized operators. As a matter of fact, it follows from an argument of Araki that it cannot hold in more than two spacetime dimensions, cf. [10, Sec. 2]. It would therefore be desirable to establish less stringent conditions which still imply that the relative commutant fixed by a given inclusion of von Neumann algebras is non–trivial. The present results seem to suggest that this information is encoded in spectral properties of the corresponding modular operators, but a clarification of this point requires some further analysis.

An appropriately weakened condition which would allow one to establish the existence of local operators in non–local algebras also in higher dimensions would have several interesting applications. This existence problem was recently met in the context of theories of massless particles with infinite spin [33], for example. It also appears in the algebraic approach to the construction of theories of particles with anyonic statistics [32] and the construction of nets of wedge algebras from information on the modular data [6, 14, 25, 39, 40]. A solution of this problem would thus be a major step in the algebraic approach to constructive problems in local quantum physics.

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# References


[1] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Exact formfactors in integrable quantum field theories: The sine-Gordon model, Nucl. Phys. B **538** (1999) 535 [arXiv:hep-th/9805185].

[2] H. Babujian and M. Karowski, Exact form factors in integrable quantum field theories: The sine-Gordon model. II, Nucl. Phys. B **620** (2002) 407 [arXiv:hep-th/0105178].

[3] B. Berg, M. Karowski and P. Weisz, Construction of Green functions from an exact S matrix, Phys. Rev. D **19** (1979) 2477.

[4] J.J. Bisognano and E.H. Wichmann, On the duality condition for a hermitian scalar field, J. Math. Phys. **16** (1975) 985.

[5] H.-J. Borchers, The CPT theorem in two-dimensional theories of local observables, Commun. Math. Phys. **143** (1992) 315.

[6] H.-J. Borchers, On revolutionizing quantum field theory with Tomita's modular theory, J. Math. Phys. **41** (2000) 3694.

[7] H. J. Borchers, D. Buchholz and B. Schroer, Polarization-free generators and the S-matrix, Commun. Math. Phys. **219** (2001) 125 [arXiv:hep-th/0003243].

[8] J. Bros, A proof of Haag-Swieca's compactness property for elastic scattering states, Commun. Math. Phys. **237** (2003) 289.

[9] R. Brunetti, D. Guido and R. Longo, Modular localization and Wigner particles, Rev. Math. Phys. **14** (2002) 759 [arXiv:math-ph/0203021].

[10] D. Buchholz, Product states for local algebras, Commun. Math. Phys. **36** (1974) 287.

[11] D. Buchholz, C. D'Antoni and K. Fredenhagen, The universal structure of local algebras, Commun. Math. Phys. **111** (1987) 123.

[12] D. Buchholz, C. D'Antoni and R. Longo, Nuclear maps and modular structures. 1: General properties, J. Funct. Anal. **88** (1990) 233.

[13] D. Buchholz, C. D'Antoni and R. Longo, Nuclear maps and modular structures. 2. Applications to quantum field theory, Commun. Math. Phys. **129** (1990) 115.

[14] D. Buchholz, O. Dreyer, M. Florig and S. J. Summers, Geometric modular action and spacetime symmetry groups, Rev. Math. Phys. **12** (2000) 475 [arXiv:math-ph/9805026].

[15] D. Buchholz and P. Jacobi, On the nuclearity condition for massless fields, Lett. Math. Phys. **13** (1987) 313.

[16] D. Buchholz and E.H. Wichmann, Causal independence and the energy level density of states in local quantum field theory, Commun. Math. Phys. **106** (1986) 321.

[17] O.A. Castro-Alvaredo and A. Fring, Mutually local fields from form factors, Int. J. Mod. Phys. B **16** (2002) 1915 [arXiv:hep-th/0112097].

[18] C. D'Antoni and K. Fredenhagen, Charges in space - like cones, Commun. Math. Phys. **94** (1984) 537.

[19] C. D'Antoni and R. Longo, Interpolation by type I infinity factors and the flip automorphism, J. Funct. Anal. **51** (1983) 361.

[20] J. Dixmier and O. Marechal, Vecteurs totalisateurs dùne algèbre de von Neumann, Commun. Math. Phys. **22** (1971) 44.

[21] S. Doplicher and R. Longo, Standard and split inclusions of von Neumann algebras, Commun. Math. Phys. **75** (1984) 493.

[22] M. Florig, On Borchers' Theorem, Lett. Math. Phys. **46** (1998) 289.

[23] R. Haag, *Local Quantum Physics*, Springer 1996.

[24] U. Haagerup, Connes' bicentralizer problem and uniqueness of the injective factor of type $\mathrm{III}_1$, Acta Math. **158** (1987), 95.

[25] R. Kähler and H.W. Wiesbrock, Modular theory and the reconstruction of four-dimensional quantum field theories, J. Math. Phys. **42** (2001) 74.

[26] M. Karowski and P. Weisz, Exact form-factors in (1+1)-dimensional field theoretic models with soliton behavior, Nucl. Phys. B **139** (1978) 455.

[27] G. Lechner, Polarization-free quantum fields and interaction, Lett. Math. Phys. **64** (2003) 137 [arXiv:hep-th/0303062].

[28] R. Longo, Notes On Algebraic Invariants For Noncommutative Dynamical Systems, Commun. Math. Phys. **69** (1979) 195.

[29] M. Müger, Superselection structure of massive quantum field theories in 1+1 dimensions, Rev. Math. Phys. **10** (1998) 1147 [arXiv:hep-th/9705019].

[30] M. Müger, Superselection structure of quantum field theories in 1+1 dimensions. Doctoral dissertation. University of Hamburg, 1997. DESY-preprint 97-073.

[31] J. Mund, The Bisognano-Wichmann theorem for massive theories, Annales Henri Poincare **2** (2001) 907 [arXiv:hep-th/0101227].

[32] J. Mund, Modular localization of massive particles with any spin in D = 2+1, J. Math. Phys. **44** (2003) 2037 [arXiv:hep-th/0208195].

[33] J. Mund, B. Schroer and J. Yngvason, String-localized quantum fields from Wigner representations, arXiv:math-ph/0402043.

[34] S. Sakai, *C*–Algebras and W*–Algebras*, Springer 1971.

[35] B. Schroer, Modular localization and the bootstrap-formfactor program, Nuclear Phys. **B 499** (1997) 547. [arXiv:hep-th/9702145].

[36] B. Schroer, Modular wedge localization and the d = 1+1 formfactor program, Annals Phys. **275** (1999) 190 [arXiv:hep-th/9712124].

[37] B. Schroer and H.W. Wiesbrock, Modular constructions of quantum field theories with interactions, Rev. Math. Phys. **12** (2000) 301 [arXiv:hep-th/9812251].

[38] F.A. Smirnov, *Formfactors in completely integrable models of quantum field theory*, Advanced Series in Mathematical Physics, 14. World Scientific 1992.

[39] H.W. Wiesbrock, Modular inclusions and intersections of algebras in QFT. Operator algebras and quantum field theory (Rome, 1996), 609–620, Internat. Press, Cambridge, MA, 1997.

[40] H.W. Wiesbrock, Modular intersections of von-Neumann-algebras in quantum field theory, Commun. Math. Phys. **193** (1998) 269.