Towards the construction of quantum field theories from a factorizing S-matrix*

Gandalf Lechner
Institut für Theoretische Physik, Universität Göttingen,
37077 Göttingen, Germany
e-mail: lechner@physik.uni-goe.de

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Abstract

Starting from a given factorizing S-matrix S in two space-time dimensions, we review a novel strategy to rigorously construct quantum field theories describing particles whose interaction is governed by S. The construction procedure is divided into two main steps: Firstly certain semi-local Wightman fields are introduced by means of Zamolodchikov's algebra. The second step consists in proving the existence of local observables in these models.
As a new result, an intermediate step in the existence problem is taken by proving the modular compactness condition for wedge algebras.

1 Introduction

In quantum field theory, the rigorous construction of models with non-trivial interaction is one of the most challenging open problems. Although collision theory has been established a long time ago and the calculation of the scattering matrix is well understood, little is known about the inverse problem, *i.e.* the reconstruction of interacting models from a given S-matrix. The only situation in which certain steps of such a reconstruction have been carried out is the class of factorizing S-matrices on two-dimensional Minkowski space, which correspond to scattering processes in which the particle number and momenta are conserved. This issue is usually taken up in the framework of the so-called formfactor program [2, 3, 36], which aims at the construction of local quantum field theories corresponding to factorizing S-matrices by determining expectation values of local operators in scattering states. In spite of the interesting results that have been obtained for many S-matrices, the final construction of interacting Wightman fields has not been achieved up to now.

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In the present paper, we shall review a novel approach to this construction problem, which has been initiated by Schroer in the last few years (see also the contribution of B. Schroer to these conference proceedings [34]). As it mainly uses the framework of local quantum physics [1, 19] instead of Wightman theory, we will term this new approach "algebraic" as opposed to the more field theoretic concepts of the formfactor program. The starting point of the algebraic approach was Schroer's insight that for the family of factorizing S-matrices describing a single type of massive, scalar particles, certain field operators arising from Zamolodchikov's algebra [41] (in which the given factorizing S-matrix S is encoded) can be interpreted as being localized in wedge-shaped regions of Minkowski space [33]. Subsequently, the understanding of these wedge-local fields was deepened in [35] and [22], and connection to the algebraic formulation of quantum field theory was made by investigating the von Neumann algebras generated by them [14]. The construction of the wedge-local fields and their corresponding operator algebras will be reviewed in section two.

The second step of the algebraic program is devoted to exhibiting *local* observables. Compared to the formfactor program, where the aim is an explicit construction of local field operators, the algebraic approach focusses on the question of *existence* of local operators, which can be phrased in terms of the aforementioned wedge algebras. In [14], the modular nuclearity condition [10, 11] for wedge algebras was put forward as a sufficient condition for the existence of local observables. This condition was then shown to be fulfilled in specific models in [14, 23]. These subjects will be discussed in section three.

Whereas the subsequent sections two and three have the character of a review, we will prove a new result in section four, already announced in [14]. In a specific class of models with factorizing S-matrices, the modular compactness criterion for wedge algebras will be verified, thus taking a further step towards proving the existence of local observables. Regarding the occasion of this conference, we emphasize the relation of our compactness proof to the work of J. Bros [7] on the 'Haag-Swieca compactness property' [20]. Inspired by his strategy, we will also employ techniques of complex analysis in section four.

The article ends with a short summary in section five.

We conclude this introductory section by stating our assumptions: In the spirit of the inverse scattering approach, our construction begins with the specification of the particle content of the theory and the S-matrix. We deal here only with a single species of scalar particles of mass $m > 0$. It will be convenient to parametrize the upper mass shell by the rapidity θ via

$$p(\theta) \;=\; m\begin{pmatrix}\cosh\theta\\ \sinh\theta\end{pmatrix}. \tag{1.1}$$

In this variable, the physical sheet of the complex energy plane is transformed to the horizontal strip $S(0,\pi) := \{\zeta \in \mathbb{C} : 0 < \mathrm{Im}\zeta < \pi\}$.[1]

[1] More generally, we will use the notation $S(a,b) = \{\zeta \in \mathbb{C} : a < \mathrm{Im}\zeta < b\}$ in the following.

As mentioned before, the S-matrix is assumed to be of the factorizing type. This implies that it can be described by means of a single function S_2, called the *scattering function* in the following, which is related to two-particle S-matrix elements by

$$ {}_{\mathrm{out}}\langle\theta_1, \theta_2|\theta_1, \theta_2\rangle_{\mathrm{in}} \;=\; S_2(|\theta_1 - \theta_2|)\,, \tag{1.2}$$

and is required to satisfy the following conditions:

1. $S_2 : \overline{S(0, \pi)} \to \mathbb{C}$ is continuous and bounded, and analytic on $S(0, \pi)$.

2. For real θ one has

$$ S_2(\theta)^{-1} = S_2(-\theta) = \overline{S_2(\theta)} = S_2(\theta + i\pi)\,. \tag{1.3}$$

By excluding poles of S_2 in the strip $S(0, \pi)$, the first condition characterizes models without bound states. The equations summarized in (1.3) arise from the requirements of unitarity, crossing symmetry and hermitian analyticity for the corresponding S-matrix (*cf.*, for example, the review in [16, p. 46] and the references cited there), and put strong constraints on the possible form of the function S_2. Indeed, the general solution of (1.3) is quite explicitely known [27]. We note here the existence of two particularly simple solutions, namely $S_2(\theta) = \pm 1$, which will be discussed in some detail. A more generic scattering function is given by

$$ S_2(\theta) \;=\; \frac{1 + ig \sinh\theta}{1 - ig \sinh\theta}\,, \tag{1.4}$$

where $g > 0$ is some constant. It interpolates between the preceding solutions in the limit of small and large g, respectively. We also note that products of scattering functions again satisfy (1.3).

2 Wedge-local fields

2.1 Hilbert space and Zamolodchikov's algebra

The starting point of the construction of the wedge-local fields is Zamolodchikov's algebra[2] [41], which is a basic ingredient in the context of factorizing S-matrices. We do not deal with the abstract algebra here, but rather with a particular representation of it on a conveniently chosen Hilbert space $\mathcal{H}$, which we define first. For details we refer the reader to [24, 22].

In view of the assumptions on the particle spectrum made above, a reasonable choice for the one particle Hilbert space is $\mathcal{H}_1 := L^2(\mathbb{R}, d\theta)$, the space of square integrable functions over the upper mass shell of mass m. The n-particle space $\mathcal{H}_n$ is defined as a particular subspace of the (unsymmetrized) n-fold tensor product $\mathcal{H}_1^{\otimes n}$; namely its elements are wavefunctions $\psi_n \in L^2(\mathbb{R}^n, d^n\theta)$ which satisfy the symmetry relations

$$ \psi_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n) \;=\; S_2(\theta_k - \theta_{k+1}) \cdot \psi_n(\theta_1, ..., \theta_k, \theta_{k+1}, ..., \theta_n) \tag{2.1}$$

[2]Sometimes also called Zamolodchikov-Faddeev algebra

for $k = 1, ..., n-1$. Here S_2 is the scattering function corresponding to the S-matrix we are considering. The full Hilbert space of the theory is

$$\mathcal{H} \;:=\; \bigoplus_{n=0}^{\infty} \mathcal{H}_n \,, \tag{2.2}$$

where we have put $\mathcal{H}_0 := \mathbb{C} \cdot \Omega$ to denote the zero particle space containing the vacuum unit vector Ω. For the special scattering functions $S_2 = 1$ and $S_2 = -1$, $\mathcal{H}$ coincides with the symmetric and antisymmetric Fock space over $\mathcal{H}_1$, respectively. But in general we deal with a "twisted" Fock space with rapidity dependent symmetry structure.

On $\mathcal{H}$ we have a positive energy representation U of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$. If $(x, \lambda) \in \mathcal{P}_+^\uparrow$ denotes the transformation consisting of a boost with rapidity parameter $\lambda \in \mathbb{R}$ and a subsequent translation along $x \in \mathbb{R}^2$, $U(x, \lambda)$ is defined as, $\psi_n \in \mathcal{H}_n$,

$$(U(x,\lambda)\psi_n)(\theta_1, ..., \theta_n) \;=\; e^{i\sum_{k=1}^n p(\theta_k)x} \cdot \psi_n(\theta_1 - \lambda, ..., \theta_n - \lambda) \,. \tag{2.3}$$

In the following, we will also use the shorthand notation $U(x) := U(x, 0)$ for pure translations.

The creation and annihilation operators familiar from symmetric and antisymmetric Fock space have their counterparts on $\mathcal{H}$. These operator valued distributions will be denoted $z(\theta)$ and $z^\dagger(\theta) = z(\theta)^*$, and are defined by

$$(z(\theta)\psi_n)(\theta_1, ..., \theta_n) \;=\; \sqrt{n} \cdot \psi_n(\theta, \theta_1, ..., \theta_n) \tag{2.4}$$

and by taking the adjoint on $\mathcal{H}$. This definition yields a representation of Zamolodchikov's algebra on $\mathcal{H}$, *i.e.* $z(\theta)$, $z^\dagger(\theta)$ satisfy the exchange relations

$$z^\dagger(\theta_1) z^\dagger(\theta_2) \;=\; S_2(\theta_1 - \theta_2) z^\dagger(\theta_2) z^\dagger(\theta_1) \,, \tag{2.5}$$

$$z(\theta_1) z^\dagger(\theta_2) \;=\; S_2(\theta_2 - \theta_1) z^\dagger(\theta_2) z(\theta_1) + \delta(\theta_1 - \theta_2) \cdot 1 \,. \tag{2.6}$$

We will also write $z(\psi) = \int d\theta\, z(\theta)\psi(\theta)$, $z^\dagger(\psi) = \int d\theta\, z^\dagger(\theta)\psi(\theta)$, for wave functions $\psi \in \mathcal{H}_1$. Note that with these conventions, $z(\psi)^* = z^\dagger(\overline{\psi})$.

2.2 Wedge locality

With the help of the creation and annihilation operators $z^\dagger(.)$ and $z(.)$, a scalar quantum field ϕ can be defined on (a dense domain in) $\mathcal{H}$ in a manner analogous to the definition of the free field on symmetric Fock space. For $f \in \mathscr{S}(\mathbb{R}^2)$, we consider the restrictions of the Fourier transform of this function to the upper and lower mass shell, parametrized by the rapidity:

$$f^\pm(\theta) \;:=\; \frac{1}{2\pi} \int d^2x\, f(x) e^{\pm ip(\theta)x} \,, \tag{2.7}$$

and set

$$\phi(f) \;:=\; z^\dagger(f^+) + z(f^-) \,, \tag{2.8}$$

which is a well-defined operator on the dense subspace $\mathcal{D} \subset \mathcal{H}$ of vectors of finite particle number. In the case of the scattering function $S_2 = 1$, this definition yields the well-known free scalar field. But also for different scattering functions, ϕ has many properties in common with a free field.

Proposition 2.1 **[22]** *The field operator $\phi(f)$ has the following properties:*

1. *$\phi(f)$ is defined on $\mathcal{D}$ and leaves this space invariant.*
2. *For $\psi \in \mathcal{D}$ one has*
$$\phi(f)^*\psi = \phi(\overline{f})\psi. \tag{2.9}$$
All vectors in $\mathcal{D}$ are entire analytic for $\phi(f)$. If $f \in \mathscr{S}(\mathbb{R}^2)$ is real, $\phi(f)$ is essentially self-adjoint on $\mathcal{D}$.
3. *ϕ is a solution of the Klein-Gordon equation: For every $f \in \mathscr{S}(\mathbb{R}^2)$, $\psi \in \mathcal{D}$ one has*
$$\phi((\Box + m^2)f)\psi = 0\,. \tag{2.10}$$
4. *$\phi(f)$ transforms covariantly under the representation U of $\mathcal{P}_+^\uparrow$, cf. (2.3):*
$$U(g)\phi(f)U(g)^{-1} = \phi(f_g), \quad f_g(x) = f(g^{-1}x), \quad g \in \mathcal{P}_+^\uparrow\,. \tag{2.11}$$
5. *The vacuum Ω is locally cyclic for the field ϕ. More precisely, given any open subset $\mathcal{O} \subset \mathbb{R}^2$, the subspace*
$$D_{\mathcal{O}} := \text{span}\{\phi(f_1)\cdots\phi(f_n)\Omega \,:\, f_k \in \mathscr{S}(\mathcal{O}),\, n \in \mathbb{N}_0\} \tag{2.12}$$
is dense in $\mathcal{H}$.

In spite of these pleasant properties of the field operator, a simple calculation shows that ϕ is local if and only if $S_2 = 1$. As locality is one of the fundamental principles in quantum field theory, the generically non-local field operators $\phi(f)$ cannot be interpreted as the basic physical fields of our model, but rather as an auxiliary tool in the construction of the theory: They are polarization-free generators in the sense of [6].

To clarify the role of the field ϕ, we consider subsets W of $\mathbb{R}^2$ called *wedges*, which are the Poincaré transforms of the so-called left wedge
$$W_L \;:=\; \{x \in \mathbb{R}^2 \,:\, |x_0| + x_1 < 0\}\,. \tag{2.13}$$
As W_L is invariant under boosts, any wedge has the form $W = W_L + x$ or $W = W_R + x$ for some $x \in \mathbb{R}^2$, where $W_R = -W_L = W_L'$ is the right wedge. The set of wedges will be denoted by $\mathcal{W}$.

Following Schroer and Wiesbrock [35], we address the question whether it is possible to consistently interpret the field ϕ as being localized in a wedge region, say in W_L for the sake of concreteness. Put differently, we take
$$\mathcal{A}(W_L) \;:=\; \{e^{i\phi(f)} \,:\, f \in \mathscr{S}_{\mathbb{R}}(W_L)\}'' \tag{2.14}$$

as the von Neumann algebra generated by the observables in W_L and look for a map $\mathcal{W} \ni W \longmapsto \mathcal{A}(W)$ of wedge regions to von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}(W_L)$ is given by (2.14) and the following standard properties [1, 19] hold: $(W, W_1, W_2 \in \mathcal{W})$

1. $\mathcal{A}(W_1) \subset \mathcal{A}(W_2)$ for $W_1 \subset W_2$ (Isotony)
2. $U(g)\mathcal{A}(W)U(g)^* = \mathcal{A}(gW)$, $g \in \mathcal{P}_+^\uparrow$ (Covariance)
3. $\mathcal{A}(W') \subset \mathcal{A}(W)'$ (Wedge-Locality)
4. Ω is cyclic for each $\mathcal{A}(W)$ (Reeh-Schlieder property)

Such a map $W \longmapsto \mathcal{A}(W)$ will be called a *wedge-local covariant net*. Within the present context one obtains a net by setting

$$\mathcal{A}(W_R) := \mathcal{A}(W_L)', \tag{2.15}$$
$$\mathcal{A}(W+x) := U(x)\mathcal{A}(W)U(x)^*, \qquad x \in \mathbb{R}^2,\ W \in \mathcal{W}, \tag{2.16}$$

where the prime denotes taking the commutant in $\mathcal{B}(\mathcal{H})$. Whereas the first three properties (isotony, covariance, and wedge locality) follow immediately from the definitions (2.14-2.16), the cyclicity of Ω for $\mathcal{A}(W_R)$ is not so obvious – Proving it is equivalent to showing that ϕ can be interpreted as being localized in W_L.

Proposition 2.2 [22, 14]
The correspondence $W \longmapsto \mathcal{A}(W)$ defined in (2.14,2.15,2.16) is a wedge-local covariant net. In particular, Ω is cyclic and seperating for each $\mathcal{A}(W)$, $W \in \mathcal{W}$. Moreover, $\mathcal{A}(W') = \mathcal{A}(W)'$, i.e. wedge duality holds.

The cyclicity of Ω can be proven by considering the antilinear operator J,

$$(J\psi)_n(\theta_1, ..., \theta_n) := \overline{\psi_n(\theta_n, ..., \theta_1)}, \tag{2.17}$$

which can be adjoined to the representation U to obtain a representation of the proper Poincaré group $\mathcal{P}_+$. More importantly, it gives rise to a second field ϕ',

$$\phi'(f) := J\phi(f^j)J, \qquad f^j(x) := \overline{f(-x)}. \tag{2.18}$$

The "reflected field" $\phi'(f)$, can be shown to commute with $\phi(g)$, in the sense that their associated unitary groups commute, whenever $\operatorname{supp} f + W_R$ is spacelike seperated from $\operatorname{supp} g + W_L$. As the vacuum is cyclic for ϕ' as well, the cyclicity of Ω for all wedge algebras then follows.
In the next section, the modular data associated to $(\mathcal{A}(W_L), \Omega)$ will become important. As the J maps $\mathcal{A}(W_L)$ onto $\mathcal{A}(W_R)$, it can be shown to coincide with the modular conjugation of this couple. The modular group $\Delta_L^{i\lambda}$ of $(\mathcal{A}(W_L), \Omega)$ acts as expected from the Bisognano-Wichmann theorem [4, 5, 29]:

Proposition 2.3 [14]
The modular group and conjugation of $(\mathcal{A}(W_L), \Omega)$ are given by $\Delta_L^{i\lambda} = U(0, 2\pi\lambda)$ and J, respectively.

Before entering into the discussion of local observables in these models, we mention that it is possible to calculate the two-particle scattering states with the help of the wedge-local fields ϕ and ϕ', since left and right wedges can be causally seperated by translation. Because of wedge-locality, it turns out that the particles are "Bosons". It is then possible to determine the two-particle S-matrix, which is the "right" one, *i.e.* the one associated to the scattering function S_2 we started with [22]. The construction of the wedge algebras thus leads to a (wedge-local) quantum field theory of particles whose interaction is described by S_2.

3 Existence of local observables

3.1 Observables localized in a double cone

Having constructed a wedge-local quantum theory with the correct two-particle scattering states, the next step is to exhibit observables localized in *bounded* spacetime regions. Typical examples of such regions are double cones, which in two dimensions can always be realized as intersections of two opposite wedges. To fix ideas, consider the double cone

$$\mathcal{O}_x \quad := \quad W_R \cap (W_L + x), \qquad x \in W_R\,. \tag{3.1}$$

An operator A representing an observable localized in $\mathcal{O}_x$ has to commute with any observable localized in ${\mathcal{O}_x}' = W_L \cup (W_R + x)$ because of Einstein causality. Any such A is therefore an element of the algebra

$$\mathcal{A}(\mathcal{O}_x) \quad := \quad (\mathcal{A}(W_L) \vee \mathcal{A}(W_R + x))' = \mathcal{A}(W_R) \cap \mathcal{A}(W_L + x)\,, \tag{3.2}$$

the relative commutant of $\mathcal{A}(W_R{+}x)$ in $\mathcal{A}(W_R)$. We will adopt (3.2) as the definition of the algebra generated by the observables localized in $\mathcal{O}_x$ in our model. The algebras associated to translated double cones are then fixed by covariance.
The net $\mathcal{O} \longmapsto \mathcal{A}(\mathcal{O})$ of double cone algebras arising in this manner inherits the basic properties of isotony, covariance and locality from the corresponding features of the wedge net, as can be verified in a straightforward manner. But without further information on the structure of the wedge algebras, it is not clear whether the relative commutants (3.2) are non-trivial. As a physical theory should describe local observables, we would like to rule out the pathological cases in which $\mathcal{A}(\mathcal{O}_x) = \mathbb{C}\cdot 1$.

In [35], a method to construct explicitely non-trivial operators localized in $\mathcal{O}_x$ has been proposed. However, this procedure faces substantial difficulties related to the convergence of certain "perturbation" series. We will concentrate here on an existence proof without trying to give concrete expressions for local operators.

3.2 Split property and modular nuclearity

The basic idea in the approach to the existence problem proposed in [14] is the observation that the non-triviality of the relative commutant $\mathcal{A}(\mathcal{O}_x)$ (3.2) can be established if the net of wedge algebras $\mathcal{A}(W)$ has the *split property*, *i.e* if to each $x \in W_R$ there is a type I_∞ von Neumman factor $\mathcal{N}_x$ such that

$$\mathcal{A}(W_R + x) \subset \mathcal{N}_x \subset \mathcal{A}(W_R)\,. \tag{3.3}$$

In this case, the observables localized in W_L and $W_R + x$ satisfy a strong form of statistical independence (for a review, see [40]), and the algebraic structure of $\mathcal{A}(\mathcal{O}_x)$ is completely fixed. According to a result of Longo [25], the wedge algebras $\mathcal{A}(W)$ are type III_1 von Neumann factors in the present context. Using this information and the split assumption (3.3), one can establish the unitary equivalence [17, 14]

$$\mathcal{A}(\mathcal{O}_x) \cong \mathcal{A}(W_R) \otimes \mathcal{A}(W_L + x). \tag{3.4}$$

Thus the split property for the wedge algebras implies that the local algebras $\mathcal{A}(\mathcal{O}_x)$ are of type III as well, and in particular non-trivial.
In the following, the split property will be used as a sufficient condition for the non-triviality of the relative commutants (3.2)[3]. However, as the existence of an interpolating type I factor (3.3) is difficult to establish directly, one needs another condition, implying the split property, which is better manageable in concrete models. In the literature different "nuclearity" criteria for the split property have been discussed, the one which is relevant in our context being introduced in [10]. As these criteria involve the notions of nuclear maps, we recall that a bounded operator between two Banach spaces is called *nuclear* if it can be expanded into a norm convergent series of rank one operators [30]. Note that a nuclear map is in particular compact.
To formulate the relevant condition for the split property, we denote by J, Δ the modular involution and modular operator of the pair $(\mathcal{A}(W_R), \Omega)$, respectively[4], and introduce the maps

$$\begin{aligned} \Xi(x) : \mathcal{A}(W_R) &\longrightarrow \mathcal{H}, \\ A &\longmapsto \Delta^{1/4} U(x) A\Omega\,. \end{aligned} \tag{3.5}$$

Using modular theory, one easily finds

$$\|\Xi(x)A\|^2 \;=\; \langle U(x)A\Omega, \Delta^{1/2}U(x)A\Omega\rangle = \langle U(x)A\Omega, JU(x)A^*\Omega\rangle \leq \|A\|^2,$$

i.e. $\Xi(x)$ is a bounded map with $\|\Xi(x)\| \leq 1$ for any $x \in \overline{W_R}$. Based on results of [10], the following "modular nuclearity condition" has been discussed in [14].

Proposition 3.1 [14]
If $\Xi(x)$ (3.5) is nuclear, the inclusion $\mathcal{A}(W_R + x) \subset \mathcal{A}(W_R)$ is split and the local algebra $\mathcal{A}(\mathcal{O}_x)$ (3.2) is isomorphic to the unique hyperfinite type III_1 factor.

[3] Note that in two space-time dimensions the split property for wedges is a reasonable assumption since Araki's argument [9] to the effect that inclusions of wedge algebras cannot be split does not apply here because of the missing translation invariance along the edge of the wedge.

[4] Δ is connected to the earlier discussed modular operator of the left wedge by $\Delta = \Delta_L^{-1}$.

Proposition 3.1 is used as a sufficient condition for the non-triviality of the local algebras in the model theories presented in section two. If the scattering function is constant, $S_2(\theta) = \pm 1$, the structure of Zamolodchikov's algebra simplifies to the CCR- and CAR-algebra, respectively, and the estimates needed for the nuclearity proof of $\Xi(x)$ are fully under control. We have

Proposition 3.2 [14, 23]
In the models corresponding to the constant scattering functions $S_2(\theta) = 1$ and $S_2(\theta) = -1$, the maps $\Xi(x)$ (3.5) are nuclear, $x \in W_R$. In particular, the split property for wedges holds and all double cone algebras (3.2) contain non-trivial observables.

The proof for the case $S_2(\theta) = 1$ can be found in [14], where previous results obtained in [13, 15] have been used. For the case $S_2(\theta) = -1$, see [23]. In these articles, one also finds explicit bounds on the nuclear norms $\|\Xi(x)\|_1$. The case $S_2(\theta) = 1$ gives just the free scalar field in two dimensions, and the model corresponding to the scattering function $S_2(\theta) = -1$ is related to the scaling limit of the Ising model (see [3] and the references cited there).
Although the existence of local observables is well known in free field theory, the check of the modular nuclearity condition in the case $S_2(\theta) = 1$ was an important test for its value in the discussion of models with non-constant scattering functions. In view of Proposition 3.2 and the earlier mentioned fact that $S_2(\theta) = \pm 1$ may be considered as the "limiting cases" of typical non-constant scattering functions, we conjecture that $\Xi(x)$ is nuclear in the family of models considered in section two.

It was shown in [10] that whereas the nuclearity of $\Xi(x)$ is sufficient for the split property, the *compactness* of $\Xi(x)$ is a necessary condition for split. In the next section, our conjecture about the nuclearity of $\Xi(x)$ will be further substantiated by proving the compactness of this map in a wide class of models with certain typical scattering functions.

4 Modular Compactness for Wedge Algebras

In this section we concentrate on the model corresponding to the scattering function (1.4) with arbitrary constant $g > 0$, or a finite product of such functions with different values of g. In view of the general solution [27] of the constraining equations (1.3) for S_2, this is a typical example of a non-constant scattering function. The aim of this section is the proof of the following Proposition.

Proposition 4.1 *Consider the model theory corresponding to the scattering function*

$$S_2(\theta) \;:=\; \prod_{r=1}^{R} \frac{1 + ig_r \sinh\theta}{1 - ig_r \sinh\theta}, \tag{4.1}$$

where $R < \infty$ and $g_1, ..., g_R > 0$. The maps $\Xi(x)$ are compact, $x \in W_R$.

Before explaining our strategy of the proof, we make a few remarks about the maps $\Xi(x)$ and introduce some notation.
First note that it is sufficient to consider the maps $\Xi(0,s)$ corresponding to wedge inclusions of the type

$$W_R + (0,s) \;\subset\; W_R, \qquad s > 0\,. \tag{4.2}$$

As W_R is stable under boosts, a more general inclusion $W_R + x \subset W_R$, $x \in W_R$, of right wedges can be transformed to (4.2) by a velocity transformation with appropriately chosen rapidity parameter λ. Using the fact that the boosts commute with the modular operator, one easily shows $\Xi(x) = U(0,-\lambda)\Xi(0,s)\mathrm{Ad}U(0,\lambda)$, where $s = (x_1^2 - x_0^2)^{1/2} > 0$. Hence $\Xi(x)$, $x \in W_R$, is nuclear (compact) if and only if $\Xi(0,s)$, $s > 0$, is nuclear (compact). In the case of nuclear maps, $\|\Xi(x)\|_1 = \|\Xi(0,(x_1^2 - x_0^2)^{1/2})\|_1$. For this reason, we will consider in the following only inclusions of the form (4.2), and use the shorthand notation $\Xi(s) := \Xi(0,s)$.

It will be useful to consider, as a generalization of (3.5), the maps

$$\Xi^\alpha(s)\mathcal{A}(W_R) \;\longrightarrow\; \mathcal{H}\,, \quad \Xi^\alpha(s)A := \Delta^\alpha U(s)A\Omega\,, \qquad 0 \le \alpha \le \tfrac{1}{2}, \tag{4.3}$$

and we adopt the convention to suppress the upper index for the "canonical" value $\alpha = \frac{1}{4}$. Furthermore, we introduce the n-particle projections[5]

$$\Xi_n^\alpha(s) \;:=\; P_n \Xi^\alpha(s)\,, \tag{4.4}$$

where $P_n \in \mathcal{B}(\mathcal{H})$ denotes the orthogonal projection onto the n-particle subspace $\mathcal{H}_n$.

In the proof of Proposition 4.1, the maps $\Xi_n(s)$ will be shown to be nuclear by estimating the "size" of their images in $\mathcal{H}_n$. This is achieved by exploiting certain analytic properties of the n-particle rapidity wavefunctions

$$\psi_n^s \;:=\; P_n U(0,s)A\Omega\,, \qquad A \in \mathcal{A}(W_R), \tag{4.5}$$

which are considered as elements of $L^2(\mathbb{R}^n, d^n\boldsymbol{\theta})$ and constitute our main objects of interest in the following. (For a different compactness proof based on the techniques of complex analysis, see [7].)
From modular theory and the Bisognano-Wichmann property (as specified in Proposition 2.3) it is known that

$$\lambda \longmapsto (\Delta^{-i\lambda/2\pi}\psi_n^s)(\theta_1, ..., \theta_n) = \psi_n^s(\theta_1 - \lambda, ..., \theta_n - \lambda) \tag{4.6}$$

is a strongly analytic function in the strip $S(0,\pi)$, and $\|\Delta^{\lambda/2\pi}\psi_n^s\| \le \|A\|$, $0 \le \lambda \le \pi$. In particular, the vectors in the image of $\Xi_n(s)$ are given by the functions

$$(\theta_1, ..., \theta_n) \longmapsto (\Delta^{1/4}\psi_n^s)(\theta_1, ..., \theta_n) \;=\; \psi_n^s(\theta_1 - \tfrac{i\pi}{2}, ..., \theta_n - \tfrac{i\pi}{2}), \tag{4.7}$$

which have an analytic continuation (in the sense of distributions) in the "center of mass rapidity" $n^{-1}\cdot(\theta_1 + ... + \theta_n)$ to the strip $S(-\frac{\pi}{2}, \frac{\pi}{2})$. Furthermore, the L^2-bound

[5] Note that the modular operator Δ can be restricted to the n-particle space $\mathcal{H}_n$.

of the continuation is uniformly bounded over this strip and the convergence to the boundary values is valid in the norm topology of $\mathcal{H}_n$.
The main idea in the proof of Proposition 4.1 consists in the observation that in the models at hand, the wavefunctions (4.5) enjoy considerably stronger analytic properties, namely they are holomorphic, as functions of n complex variables, in a certain tube domain. More precisely, we find:

Lemma 4.2 *Let $A \in \mathcal{A}(W_R)$, $s > 0$, $n \in \mathbb{N}_0$ and ψ_n^s as in (4.5). There exists a constant $\alpha > 0$ (independent of A and s, but dependent on n) such that*

a) $\Delta^\alpha \psi_n^s$ is analytic in the tube

$$\mathcal{T}_n(\alpha) \;:=\; \{\boldsymbol{\zeta} \in \mathbb{C}^n \,:\, -2\pi\alpha < \mathrm{Im}(\zeta_k) < 2\pi\alpha,\; k = 1, ..., n\}. \tag{4.8}$$

b) For any $\boldsymbol{\lambda} \in \,]-2\pi\alpha, 2\pi\alpha\,[^{\times n}$,

$$\mathbb{R}^n \ni \boldsymbol{\theta} \longmapsto (\Delta^\alpha \psi_n^s)(\boldsymbol{\theta} + i\boldsymbol{\lambda}) \tag{4.9}$$

is in $L^2(\mathbb{R}^n, d^n\boldsymbol{\theta})$, with norm bounded by $K \cdot \|A\|$, where K depends on α, s and n, but is independent of A and $\boldsymbol{\lambda}$. Moreover, $\lim\limits_{|\boldsymbol{\theta}|\to\infty} |\psi_n^s(\boldsymbol{\theta} + i\boldsymbol{\lambda})| = 0$.

c) $(\Delta^\alpha \psi_n^s)(\boldsymbol{\zeta})$ converges strongly to its boundary values at $\mathrm{Im}(\zeta_k) = \pm 2\pi\alpha$.

Here we introduced the convention to denote vectors in $\mathbb{C}^n$ or $\mathbb{R}^n$ by bold face letters $\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\theta}$, and their components by $\zeta_k, \lambda_k, \theta_k$, $k = 1, ..., n$. Note in particular that by considering the limit $\mathrm{Im}(\zeta_k) \to 2\pi\alpha$, $k = 1, ..., n$, the wavefunctions ψ_n^s (4.5) are recovered as a (strong) boundary value of the analytic function $\Delta^\alpha \psi_n^s$.
The constants α and K appearing in Lemma 4.2 specify the size of the tube $\mathcal{T}_n(\alpha)$ in which $\Delta^\alpha \psi_n^s$ is analytic and its bound in that region, respectively. They depend on the scattering function S_2 (4.1) under consideration, *i.e.* on the parameters $g_1, ..., g_R$. This dependence will be made explicit in the proof of Lemma 4.2, which is based on wedge locality and the symmetry properties (2.1) of n-particle functions. However, it will require the discussion of some technical points. We therefore postpone it and and rather begin by showing how Lemma 4.2 can be used to derive estimates on the nuclear norms of $\Xi_n(s)$.

Lemma 4.3 *The maps $\Xi_n(s)$ are nuclear, $s > 0$.*

Proof: In view of the definition of the translations (2.3), we have, $\boldsymbol{\theta} \in \mathbb{R}^n$,

$$\begin{aligned}
(\Delta^\alpha \psi_n^s)(\boldsymbol{\theta}) &= (\Delta^\alpha U(0, \tfrac{s}{2}) \psi_n^{\frac{s}{2}})(\boldsymbol{\theta}) \\
&= e^{-im\frac{s}{2}\sum_{k=1}^n \mathrm{sh}(\theta_k - 2\pi\alpha\, i)} (\Delta^\alpha \psi_n^{\frac{s}{2}})(\boldsymbol{\theta}) \\
&= \prod_{k=1}^n e^{-i\frac{ms}{2}\cos(2\pi\alpha)\,\mathrm{sh}\theta_k} e^{-\frac{ms}{2}\sin(2\pi\alpha)\,\mathrm{ch}\theta_k} \cdot (\Delta^\alpha \psi_n^{\frac{s}{2}})(\boldsymbol{\theta}).
\end{aligned} \tag{4.10}$$

The strongly decreasing factor appearing here allows us to conclude the nuclearity of $\Xi_n(s)$ by application of Cauchy's integral formula: Let $\boldsymbol{\theta} \in \mathbb{R}^n$ and consider a polydisc $\mathscr{D} \subset \mathcal{T}_n(\alpha)$ with $\boldsymbol{\theta} \in \mathscr{D}$. As $\Delta^\alpha \psi_n^s$ is holomorphic in $\mathcal{T}_n(\alpha)$,

$$(\Delta^\alpha \psi_n^s)(\boldsymbol{\theta}) \;=\; \frac{1}{(2\pi i)^n} \oint_{\partial \mathscr{D}} d^n \boldsymbol{\zeta}' \, \frac{(\Delta^\alpha \psi_n^s)(\boldsymbol{\zeta}')}{\prod_{k=1}^n (\zeta_k' - \theta_k)} \,.$$

Because of the decrease properties of $\Delta^\alpha \psi_n^s$ in real directions, and the good convergence to its boundary values, as specified in Lemma 4.2 b),c), we can deform the contour of integration to the boundary of $\mathcal{T}_n(\alpha)$ and get

$$\begin{aligned} (\Delta^\alpha \psi_n^s)(\boldsymbol{\theta}) \;=\; & \frac{1}{(2\pi i)^n} \sum_{\boldsymbol{\varepsilon}} \int_{\mathbb{R}^n} d^n \boldsymbol{\theta}' \prod_{k=1}^n \frac{\varepsilon_k}{(\theta_k' - \theta_k - i \cdot 2\pi\alpha\varepsilon_k)} \\ & \times\; (\Delta^\alpha \psi_n^s)(\theta_1' - i \cdot 2\pi\alpha\varepsilon_1, ..., \theta_n' - i \cdot 2\pi\alpha\varepsilon_n). \end{aligned} \tag{4.11}$$

The summation extends over the 2^n terms parametrized by $\varepsilon_1, ..., \varepsilon_n = \pm 1$. Consider the integral operators $T_{s,\alpha}^\pm \in \mathcal{B}(L^2(\mathbb{R}, d\theta))$ which are defined by their kernels

$$T_{s,\alpha}^\pm(\theta, \theta') \;:=\; \pm \frac{1}{i\pi} \frac{e^{-\frac{ms}{2}\sin(2\pi\alpha)\,\mathrm{ch}\theta}}{\theta' - \theta \mp 2\pi i \alpha}, \tag{4.12}$$

and the unitary operator $M_{s,\alpha}$ multiplying with $e^{-i\frac{ms}{2}\cos(2\pi\alpha)\mathrm{sh}\theta}$. With these definitions, inserting (4.10) into (4.11) yields

$$\Xi_n^\alpha(s)A = \Delta^\alpha \psi_n^s \;=\; 2^{-n} \sum_{\boldsymbol{\varepsilon}} {M_{s,\alpha}}^{\otimes n} (T_{s,\alpha}^{\varepsilon_1} \otimes ... \otimes T_{s,\alpha}^{\varepsilon_n}) \Delta^\alpha \psi_{n,\boldsymbol{\varepsilon}}^{\frac{s}{2}} \,, \tag{4.13}$$

where we used the shorthand notation $\psi_{n,\boldsymbol{\varepsilon}}^{\frac{s}{2}}(\boldsymbol{\theta}) = \psi_n^{\frac{s}{2}}(\boldsymbol{\theta} - 2\pi\alpha i \cdot \boldsymbol{\varepsilon})$. It is now important to note that $T_{s,\alpha}^\pm$ are trace class operators on $L^2(\mathbb{R}, d\theta)$ [14]. (This can be seen by Fourier transforming the kernels (4.12) in the variable θ', and the applying standard techniques to split the resulting operators into a product of two Hilbert Schmidt maps [39, Thm. XI.21].) Note also that $T_{s,\alpha}^+$ and $T_{s,\alpha}^-$ are unitarily equivalent, the equivalence being given by the operator V, $(V\psi)(\theta) := \psi(-\theta)$. Hence they have the same trace norm $\|T_{s,\alpha}^+\|_1$.

According to Lemma 4.2 b), $\|\psi_{n,\boldsymbol{\varepsilon}}^{\frac{s}{2}}\| \leq K\|A\|$ for each $\boldsymbol{\varepsilon}$ occuring in the above summation. Put differently, $A \longmapsto \Delta^\alpha \psi_{n,\boldsymbol{\varepsilon}}^{\frac{s}{2}}$ is bounded as a linear map between the Banach spaces $\mathcal{A}(W_R)$ and $\mathcal{H}_n$, with norm dominated by K. As $M_{s,\alpha}$ is unitary, this implies that $\Xi_n^\alpha(s)$ is a nuclear map, and as a crude bound on its nuclear norm we have

$$\|\Xi_n^\alpha(s)\|_1 \;\leq\; K\,\|T_{s,\alpha}^+\|_1^n \,. \tag{4.14}$$

To proceed from this nuclearity result to the statement in Lemma 4.3, the power of the modular operator needs to be adjusted from α to $\frac{1}{4}$. This can be achieved as in [10, Cor. 3.4]: Let $A \in \mathcal{A}(W_R)$. From modular theory we know

$$\Xi_n^\alpha(s)A \;=\; P_n \Delta^\alpha U(0,s) A\Omega = P_n \Delta^\alpha J \Delta^{1/2} U(0,s) A^*\Omega \tag{4.15}$$

$$\;=\; J P_n \Delta^{1/2-\alpha} U(0,s) A^*\Omega = J \Xi_n^{1/2-\alpha}(s) A^* \,, \tag{4.16}$$

where we used the fact that Δ and J commute with P_n. Hence $\Xi_n^{1/2-\alpha}(s)$ is nuclear, too, and $\|\Xi_n^{1/2-\alpha}(s)\|_1 = \|\Xi_n^{\alpha}(s)\|_1$. Since $X := \Delta^{1/4}(\Delta^\alpha + \Delta^{1/2-\alpha})^{-1}$ is a bounded operator with norm $\|X\| \leq \frac{1}{2}$, it follows from

$$\Xi_n(s) \;=\; X\Big(\Xi_n^{\alpha}(s) + \Xi_n^{1/2-\alpha}(s)\Big) \tag{4.17}$$

that $\Xi_n(s)$ is nuclear, with nuclear norm bounded by

$$\|\Xi_n(s)\|_1 \leq \|X\| \left(\|\Xi_n^{\alpha}(s)\|_1 + \|\Xi_n^{1/2-\alpha}(s)\|_1\right) \leq \|\Xi_n^{\alpha}(s)\|_1 \,. \tag{4.18}$$

This completes the proof of the Lemma. □

The proof of Proposition 4.1 now follows as a short corollary of Lemma 4.3 [14]. Let $A \in \mathcal{A}(W_R)$. Since

$$(\Xi_n(s)A)(\boldsymbol{\theta}) = \prod_{k=1}^{n} e^{-ms\cosh\theta_k} \cdot (\Xi_n(0)A)(\boldsymbol{\theta}) \leq e^{-msn} \cdot (\Xi_n(0)A)(\boldsymbol{\theta})$$

and $\|\Xi_n(0)\| \leq 1$, we have $\|\Xi_n(s)\| \leq e^{-msn}$, and hence the series $\sum_{n=0}^\infty \Xi_n(s)$ converges in the norm topology of $\mathcal{B}(\mathcal{A}(W_R), \mathcal{H})$ to $\Xi(s)$. But as nuclear maps, the $\Xi_n(s)$ are in particular compact, and the set of compact operators between two Banach spaces is norm closed [37, Thm. VI.12]. Thus $\Xi(s)$ is compact, too. □

To accomplish the existence proof for local observables, one has to show that the map $\Xi(s)$ is not only compact, but also nuclear. This amounts to proving that the series $\sum_n \Xi_n(s)$ converges not only in the operator norm $\|\cdot\|$, but also in the nuclear norm $\|\cdot\|_1$, *i.e.* better bounds than (4.14) on $\|\Xi_n(s)\|_1$ are required. Such a refined analysis will be presented elsewhere.

After having shown how analytic properties of the wavefunctions ψ_n^s lead to the nuclearity of $\Xi_n(s)$, it remains to derive these properties, *i.e.* prove Lemma 4.2. This proof will be devided into three steps: First, we consider the dependence of ψ_n^s on its first variable θ_1 only, and derive analytic properties of $\theta_1 \mapsto \psi_n^s(\theta_1, ..., \theta_n)$ by using the localization of A in W_R (Lemmata 4.4 and 4.5). This is a kind of one-particle analysis, and the form of the scattering function S_2 does not matter here. In a second step, the symmetry (2.1) is exploited to transfer these results to the other variables $\theta_2, ..., \theta_n$. Here two important properties (4.40, 4.41) of S_2 enter. Finally, the n-variable analyticity claimed in Lemma 4.2 is established by using the Malgrange-Zerner ("flat tube") theorem (*cf.*, for example, [18]).

To exploit the localization of A, it is useful to consider the time zero fields φ, π of the "left localized" field ϕ (2.8) and study expectation value functionals of the commutator of these fields with A. In rapidity space, these functionals give rise to certain analytic functions (Lemma 4.4). Their relations to the wavefunctions ψ_n^s are explained in Lemma 4.5.

The formal definitions $\varphi(x) = \phi(0, x)$, $\pi(x) = (\partial_0\phi)(0, x)$, $x \in \mathbb{R}$, can be rephrased as ($f \in \mathscr{S}(\mathbb{R})$)

$$\hat{f}(\theta) := \widetilde{f}(m \sinh\theta), \qquad \hat{f}_-(\theta) := \hat{f}(-\theta), \tag{4.19}$$

$$\varphi(f) := z^\dagger(\hat{f}) + z(\hat{f}_-), \tag{4.20}$$

$$\pi(f) := i(z^\dagger(\omega\hat{f}) - z(\omega\hat{f}_-)). \tag{4.21}$$

Here $\omega = m\cosh\theta$ is considered as an (unbounded) multiplication operator on its maximal domain in $L^2(\mathbb{R}, d\theta)$. Along the same lines as in [22, Prop. 2 (2)], one can show that these fields are localized on the left half line, *i.e.* $\varphi(f)$, $\pi(f)$ are affiliated with $\mathcal{A}(W_L)$ if $f \in \mathscr{S}(\mathbb{R}_-)$.
Choosing an $(n-1)$-particle vector $\xi_{n-1} \in \mathcal{H}_{n-1}$, an operator $A \in \mathcal{A}(W_R)$, and a translation $s > 0$, we define two linear functionals on $\mathscr{S}(\mathbb{R})$ as

$$\begin{aligned} C_s^-(f) &:= \langle \xi_{n-1}, [\varphi(f), A(0, s)]\Omega\rangle, \\ C_s^+(f) &:= \langle \xi_{n-1}, [\pi(f), A(0, s)]\Omega\rangle\,. \end{aligned} \tag{4.22}$$

The operator A is an arbitrary element of $\mathcal{A}(W_R)$, but fixed in the following. The (anti-) linear dependence of the above distributions on A and ξ_{n-1} will not be indicated in our notation.
We recall that the creation and annihilation operators $z^\dagger(.), z(.)$ satisfy the following bounds with respect to the particle number [22], familiar from free field theory ($\chi \in \mathcal{H}_1$):

$$\begin{aligned} \|z(\chi)\xi_{n-1}\| &\leq (n-1)^{1/2}\|\chi\|\|\xi_{n-1}\|, \\ \|z^\dagger(\chi)\xi_{n-1}\| &\leq n^{1/2}\|\chi\|\|\xi_{n-1}\|\,. \end{aligned} \tag{4.23}$$

Also note that for $f \in L^2(\mathbb{R}, dx)$ (we put $\|f\|_2 := (\int dx|f(x)|^2)^{1/2}$),

$$\|\omega^{1/2}\hat{f}\|^2 = \int d\theta\, m\mathrm{ch}\theta\, |\hat{f}(\theta)|^2 = \int dp\, |\widetilde{f}(p)|^2 = \|f\|_2^2\,. \tag{4.24}$$

Combining (4.23) and (4.24) with the Cauchy-Schwarz inequality and using the annihilation property of $z(\cdot)$, we obtain the bounds

$$|C_s^\pm(\omega^{\mp 1/2}f)| \leq c_n\, \|\xi_{n-1}\|\|A\| \cdot \|f\|_2, \quad c_n := \sqrt{n} + \sqrt{n-1} + 1. \tag{4.25}$$

By carrying out the same estimate for $|C_s^\pm(f)|$ and taking into account that both, $\|\hat{f}\|$ and $\|\omega\hat{f}\|$, can be dominated by certain linear combinations of Schwartz space seminorms of f, we first note $C_s^\pm \in \mathscr{S}'(\mathbb{R})$. Moreover, (4.25) shows that the distributions $f \mapsto C_s^\pm(\omega^{\mp 1/2}f)$ are regular in the sense that they are given by L^2-functions, whose norm is bounded by $c_n\|\xi_{n-1}\|\|A\|$.

In view of the localization properties of A, $\varphi(f)$ and $\pi(f)$, the support of $C_s^\pm$ is contained in the half line $[s, \infty[\, \subset \mathbb{R}_+$. Consequently, their Fourier transforms are boundary values (in the sense of distributions) of functions $\tilde{C}_s^\pm$ which are analytic in the lower half plane and polynomially bounded in imaginary direction [38, Thm. IX.16]. Taking into account supp $C_s^\pm \subset [s,\infty[$, it follows that $\tilde{C}_s^\pm$ actually decays exponentially in imaginary direction.

We now proceed to the rapidity picture by setting

$$\hat{C}_s^-(\theta) := m\cosh(\theta)\cdot \tilde{C}_s^-(m\sinh\theta), \qquad \hat{C}_s^+(\theta) := \tilde{C}_s^+(m\sinh\theta)\,. \tag{4.26}$$

Note that because of the regularity of $\omega^{\mp 1/2}\tilde{C}_s^\pm$, these are well-defined *functions*. Their properties are collected in the following Lemma.

Lemma 4.4 *Let $A \in \mathcal{A}(W_R)$, $s > 0$, and $\xi_{n-1} \in \mathcal{H}_{n-1}$. The functions $\hat{C}_s^\pm$ (4.26) corresponding to the distributions (4.22) have the following properties:*

a) $\hat{C}_s^\pm$ is the boundary value of a function analytic in the strip $S(-\pi,0)$.

b) $\hat{C}_s^\pm$ is (anti-) symmetric with respect to reflections about $\mathbb{R} - \frac{i\pi}{2}$:

$$\hat{C}_s^\pm(\theta - \tfrac{i\pi}{2} + i\mu) = \pm\hat{C}_s^\pm(-\theta - \tfrac{i\pi}{2} - i\mu), \qquad -\tfrac{\pi}{2} < \mu < \tfrac{\pi}{2}\,. \tag{4.27}$$

c) Let $0 \le \lambda \le \pi$ and $c_n = \sqrt{n} + \sqrt{n-1} + 1$. The functions

$$\mathbb{R} \ni \theta \longmapsto \hat{C}_{s,\lambda}^\pm(\theta) := \hat{C}_s^\pm(\theta - i\lambda) \tag{4.28}$$

are elements of $L^2(\mathbb{R}, d\theta)$, with norm bounded by $\|\hat{C}_{s,\lambda}^\pm\| \le c_n\|\xi_{n-1}\|\|A\|$.

d) $\overline{S(-\pi,0)} \ni \zeta \longmapsto \hat{C}_{s,\zeta}^\pm$ is continuous in the norm topology of $L^2(\mathbb{R}, d\theta)$.

e) For $\theta \in \mathbb{R}$, $0 < \lambda < \pi$,

$$|\hat{C}_s^\pm(\theta - i\lambda)| \;\le\; \sqrt{\frac{2}{\pi}}\, c_n\|\xi_{n-1}\|\|A\|\frac{e^{-\frac{ms}{2}\sin\lambda\,\cosh\theta}}{\min\{\lambda,\pi-\lambda\}^{1/2}}\,. \tag{4.29}$$

Proof: a) Recalling

$$\begin{aligned}\cosh(\theta - i\lambda) &= \cos\lambda\,\cosh\theta - i\sin\lambda\,\sinh\theta,\\ \sinh(\theta - i\lambda) &= \cos\lambda\,\sinh\theta - i\sin\lambda\,\cosh\theta,\end{aligned} \tag{4.30}$$

we see that sinh(.) maps the strip $S(-\pi,0)$ to the lower half plane. Hence $\hat{C}_s^\pm$ is analytic in $S(-\pi,0)$. b) is also a direct consequence of (4.30).
To prove c), note that on the real line the claimed bound holds in view of the former estimates on $\|\omega^{\mp 1/2}\tilde{C}_s^\pm\|$:

$$\int d\theta\,|\hat{C}_s^\pm(\theta)|^2 = \int dp\,|\omega(p)^{\mp 1/2}\tilde{C}_s^\pm(p)|^2 = \|\omega^{\mp 1/2}\tilde{C}_s^\pm\|_2^2 \le c_n^2\|\xi_{n-1}\|^2\|A\|^2.$$

But b) implies in particular $\hat{C}_s^\pm(\theta - i\pi) = \pm\hat{C}_s^\pm(-\theta)$, and hence $\hat{C}_s^\pm$ is also square integrable over the lower boundary $\mathbb{R} - i\pi$, with the same bound on its norm. As

$\tilde{C}_s^\pm$ decays exponentially in imaginary direction, we have $\hat{C}^\pm_{s,\lambda} \in L^2(\mathbb{R}, d\theta)$ for any fixed $0 < \lambda < \pi$. Moreover, the limits $\lim_{\lambda\to 0} \hat{C}^\pm_{s,\lambda}$ and $\lim_{\lambda\to\pi} \hat{C}^\pm_{s,\lambda}$ are known to hold in the sense of distributions. These facts allow for the application of a version of the three lines theorem adapted to L^2-bounds, which is proven in the appendix as Lemma A.1. The results are: $\hat{C}^\pm_{s,\lambda}$ converges in the norm topology of $L^2(\mathbb{R}, d\theta)$ as $\lambda \to 0$ or $\lambda \to \pi$, and the bound calculated on the boundary holds also for $\hat{C}^\pm_{s,\lambda}$, $0 < \lambda < \pi$. This proves the claims c) and d).
Finally, e) is a consequence of a) and c): Let $\theta \in \mathbb{R}$, $0 < \lambda < \pi$, and put $\rho := \min\{\lambda, \pi - \lambda\}$. Then the disc D_ρ with center $\theta - i\lambda$ and radius ρ is contained in the closed strip $\overline{S(-\pi, 0)}$. By the mean value theorem for analytic functions, Hölder's inequality and the norm bound given in c), we get

$$
\begin{aligned}
|\hat{C}_s^\pm(\theta - i\lambda)| &\leq \frac{1}{\pi\rho^2}\int_{D_\rho} d\theta'\, d\lambda'\, |\hat{C}_s^\pm(\theta' + i\lambda')| \\
&\leq \frac{1}{\sqrt{\pi}\rho}\left(\int_{D_\rho} d\theta'\, d\lambda'\, |\hat{C}_s^\pm(\theta' + i\lambda')|^2\right)^{1/2} \\
&\leq \frac{1}{\sqrt{\pi}\rho}\left(\int_{-\lambda-\rho}^{-\lambda+\rho} d\lambda' \int_{-\infty}^{\infty} d\theta'\, |\hat{C}_s^\pm(\theta' + i\lambda')|^2\right)^{1/2} \\
&\leq \sqrt{\frac{2}{\pi\rho}} \cdot c_n \|\xi_{n-1}\| \|A\|\,. \qquad (4.31)
\end{aligned}
$$

Taking into account the covariance of $\varphi(f)$ and the translation invariance of Ω, one obtains

$$
\begin{aligned}
C_s^-(f) &= \langle \xi_{n-1},\, [\varphi(f), U(0, \tfrac{s}{2})A(0, \tfrac{s}{2})U(0, -\tfrac{s}{2})]\Omega\rangle \\
&= \langle U(0, -\tfrac{s}{2})\xi_{n-1},\, [\varphi(f_{\frac{s}{2}}), A(0, \tfrac{s}{2})]\Omega\rangle =: C'(f_{\frac{s}{2}}),
\end{aligned}
$$

where $f_{\frac{s}{2}}(x) = f(x+\frac{s}{2})$, $x \in \mathbb{R}$. Hence $\hat{C}_s^-(\theta) = e^{-\frac{ims}{2}\mathrm{sh}\theta}\hat{C}'(\theta)$, and $\hat{C}'$ also fulfills the estimate (4.31) since $\|U(0, \pm\frac{s}{2})\| = 1$. Taking the absolute value of the exponential factor evaluated on points $\theta - i\lambda \in S(-\pi, 0)$ yields the claimed estimate (4.29) for $\hat{C}_s^-$. The argument for $\hat{C}_s^+$ is the same. □

The relation between the functions $\hat{C}_s^\pm$ and the wavefunctions ψ_n^s (4.5) is specified in the next Lemma.

Lemma 4.5 *Consider the function*[6]

$$
h_\xi^s(\zeta) := \frac{1}{2}(\hat{C}_s^-(\zeta) + i\hat{C}_s^+(\zeta)), \qquad \zeta \in \overline{S(-\pi, 0)}\,, \qquad (4.32)
$$

which depends on $\xi_{n-1} \in \mathcal{H}_{n-1}$ through the definitions (4.22) and (4.26).
h_ξ^s has the properties a), c), d), e) of the preceding Lemma. Moreover, $\theta_1 \in \mathbb{R}$,

$$
h_\xi^s(\theta_1) = \sqrt{n}\int d\theta_2 \cdots d\theta_n\, \overline{\xi_{n-1}(\theta_2, ..., \theta_n)} \cdot \psi_n^s(\theta_1, ..., \theta_n)\,. \qquad (4.33)
$$

[6]We write h_ξ^s instead of $h_{\xi_{n-1}}^s$ in order not to overburden our notation.

Proof: From the definition of h^s_ξ it follows immediately that properties a) and c)–e) of Lemma 4.4 hold. To show (4.33), let $f \in \mathscr{S}(\mathbb{R})$.

$$\begin{aligned}
\langle \overline{\hat{f}_-}, h^s_\xi \rangle &= \frac{1}{2}\int d\theta \hat{f}(-\theta)\left(\hat{C}^-_s(\theta) + i\hat{C}^+_s(\theta)\right) \\
&= \frac{1}{2}\int dp \tilde{f}(-p)\left(\tilde{C}^-_s(p) + i\omega(p)^{-1}\tilde{C}^+_s(p)\right) \\
&= \frac{1}{2}\left(C^-_s(f) + iC^+_s(\omega^{-1}f)\right) \\
&= \frac{1}{2}\langle \xi_{n-1},\, [\varphi(f) + i\pi(\omega^{-1}f), A(0,s)]\Omega\rangle \\
&= \langle \xi_{n-1},\, [z(\hat{f}_-), A(0,s)]\Omega\rangle && (4.34)\\
&= \langle \xi_{n-1},\, z(\hat{f}_-)A(0,s)\Omega\rangle && (4.35)\\
&= \langle z^\dagger(\overline{\hat{f}_-})\xi_{n-1},\, P_n U(0,s)A\Omega\rangle && (4.36)\\
&= \sqrt{n}\,\langle \overline{\hat{f}_-} \otimes \xi_{n-1},\, \psi^s_n\rangle\,. && (4.37)
\end{aligned}$$

In the last steps, we wrote the annihilation operator as a linear combination of the time zero fields in (4.34), *cf.* (4.20,4.21), used the annihilation property of $z(.)$ in (4.35), the relations of Zamolodchikov's algebra in (4.36) and the definition of $z^\dagger(.)$ in (4.37). By continuity, the above calculated relation

$$\langle \overline{\hat{f}_-}, h^s_\xi \rangle = \sqrt{n}\,\langle \overline{\hat{f}_-} \otimes \xi_{n-1},\, \psi^s_n\rangle \tag{4.38}$$

holds also if $\hat{f}_-$ is replaced by an arbitrary function in $L^2(\mathbb{R}, d\theta)$. This implies (4.33). □

For $n = 1$, Lemma 4.5 states $\psi^s_1 = h^s_\Omega$. Hence the one particle wavefunctions ψ^s_1 have the properties a), c)–e), listed in Lemma 4.4. In particular the claims of Lemma 4.2 follow if the parameters α and K appearing there are chosen as $\alpha = \frac{1}{4}$ and $K = 1$.
But for $n > 1$, only information about ψ^s_n, considered as a function of the first variable θ_1, has been obtained. To extend this to the other variables, two properties of the scattering function (4.1),

$$S_2(\theta) = \prod_{r=1}^{R} \frac{1 + ig_r \sinh\theta}{1 - ig_r \sinh\theta}, \qquad g_1, ..., g_R > 0, \tag{4.39}$$

are important, which we extract now. Firstly, we see that S_2 is a meromorphic function in the entire complex plane, with certain $(2\pi i)$-periodic sequences of poles. In particular, S_2 is analytic not only in the physical sheet $S(0,\pi)$, but also in the wider strip $S(-\kappa_g, \pi + \kappa_g)$, where $\kappa_g > 0$ is given by

$$\kappa_g := \begin{cases} \arcsin(g^{-1}) & ; \quad g > 1 \\ \frac{\pi}{2} & ; \quad 0 < g \leq 1 \end{cases}, \qquad g := \max_{r=1,\dots,R} g_r\,. \tag{4.40}$$

Furthermore, given arbitrary $\kappa \in [\,0, \kappa_g\,[$, we note that S_2 is uniformly bounded on $S(-\kappa, \pi+\kappa)$. This bound will be denoted by

$$\sigma(\kappa) := \sup\{|S_2(\zeta)| \,:\, \zeta \in \overline{S(-\kappa, \pi+\kappa)}\} < \infty, \qquad 0 < \kappa < \kappa_g\,. \tag{4.41}$$

These two features of S_2, the analyticity in the enlarged strip and the bound (4.41), will be essential in the following proof of Lemma 4.2. Furthermore, the parameters α and K appearing there, will be specified explicitely in terms of $\kappa, \sigma(\kappa)$ and n.

Proof of Lemma 4.2:
Fix $\zeta_1 = \theta_1 - i\lambda_1 \in S(-\pi, 0)$. In view of the definitions of $C_s^{\pm}$ (4.22), $\hat{C}_s^{\pm}$ (4.26) and h_ξ^s (4.32), it is clear that $\xi \longmapsto h_\xi^s(\zeta_1)$ is an anti-linear functional on $\mathcal{H}_{n-1}$, which by Lemma 4.4 e) is norm-continuous. Hence, by application of Riesz' Theorem, $h_\xi^s(\zeta_1)$ can be written as

$$h_\xi^s(\zeta_1) \;\; =: \;\; \sqrt{n} \int d\theta_2 \cdots d\theta_n\, \overline{\xi_{n-1}(\theta_2, ..., \theta_n)} \cdot \psi_n^s(\zeta_1, \theta_2, ..., \theta_n)\,, \tag{4.42}$$

where $(\theta_2, ..., \theta_n) \mapsto \psi_n^s(\zeta_1, \theta_2, ..., \theta_n)$ is a function in $\mathcal{H}_{n-1} \subset L^2(\mathbb{R}^{n-1} d^{n-1}\boldsymbol{\theta})$ which is defined by this equation. Taking ξ_{n-1} to be the characteristic function of a compact set $\mathcal{C} \subset \mathbb{R}^{n-1}$, and χ_I to be the characteristic function of a compact interval $I \subset \mathbb{R}$, it follows from

$$\sqrt{n} \left| \int_{I\times\mathcal{C}} d^n\boldsymbol{\theta}\, \psi_n^s(\theta_1 - i\lambda_1, \theta_2, ..., \theta_n) \right| \;\; = \;\; \left| \int d\theta_1 \overline{\chi_I(\theta_1)} h_\xi^s(\theta_1 - i\lambda_1) \right| < \infty$$

that $\boldsymbol{\theta} \longmapsto \psi_n^s(\theta_1 - i\lambda_1, \theta_2, ..., \theta_n) =: \psi_{n,\lambda_1}^s(\boldsymbol{\theta})$ is locally integrable and hence measurable. According to Lemma 4.4 e),

$$\int d\theta_2 \cdots d\theta_n\, |\psi_{n,\lambda_1}^s(\boldsymbol{\theta})|^2 \;\; \leq \;\; \frac{2c_n^2 \|A\|^2\, e^{-ms\sin\lambda_1 \mathrm{ch}\theta_1}}{\pi n \min\{\lambda_1, \pi - \lambda_1\}}. \tag{4.43}$$

In view of the measurability of ψ_{n,λ_1}^s, it follows from the estimate (4.43) that for fixed $\lambda_1 \in (0,\pi)$, ψ_{n,λ_1}^s is an element of $L^2(\mathbb{R}^n, d^n\boldsymbol{\theta})$. Furthermore, note that the set $\{n^{-1}c_n^2 \,:\, n \in \mathbb{N}\}$ is bounded. Hence there is a function $b(\lambda_1, s)$, independent of n and A, such that

$$\|\psi_{n,\lambda_1}^s\| \;\; \leq \;\; b(\lambda_1, s) \cdot \|A\|\,, \qquad 0 < \lambda_1 < \pi,\; s > 0\,. \tag{4.44}$$

Next we want to show that ψ_{n,λ_1}^s converges to ψ_n^s in the norm topology of $L^2(\mathbb{R}^n, d^n\boldsymbol{\theta})$ as $\lambda_1 \to 0$. To this end, we need to improve on the bound (4.44), because $b(\lambda_1, s)$ diverges for $\lambda_1 \to 0$ and for $\lambda_1 \to \pi$. Note first that as a consequence of its rapid decrease in the real direction, the Fourier transform of $(h_\xi^s)_{\lambda_1}$ is given by $\beta_1 \mapsto e^{\lambda_1\beta_1}\widetilde{h_\xi^s}(\beta_1)$, $\beta_1 \in \mathbb{R}$ (*cf.* the proof of Lemma A.1 in the appendix). In view of (4.42), this relation implies that the Fourier transform of ψ_{n,λ_1}^s is $\mathbb{R}^n \ni \boldsymbol{\beta} \mapsto e^{\lambda_1\beta_1} \cdot \widetilde{\psi_n^s}(\boldsymbol{\beta})$. Now one can proceed as in the proof of Lemma A.1 and apply Lebesgue's dominated convergence theorem to arrive at the norm limit $\psi^s_{n,\lambda_1} \to \psi^s_n$ as $\lambda_1 \to 0$. The limit $\psi^s_{n,\lambda_1} \to \psi^s_{n,\pi}$ as $\lambda_1 \to \pi$ can be established by exploiting the symmetry properties (Lemma 4.4 b)) of $\hat{C}^\pm_s$, but will not be needed here.

We now turn to the analyticity properties of the wavefunctions and consider a point $\zeta_0 \in S(-\pi, 0)$ with an appropriate curve $\mathcal{C}_0 \subset S(-\pi, 0)$ containing it in its interiour. As h^s_ξ is analytic, we have

$$0 = \oint_{\mathcal{C}_0} d\zeta\, h^s_\xi(\zeta) = \sqrt{n} \oint_{\mathcal{C}_0} d\zeta \int d\theta_2 \cdots d\theta_n\, \overline{\xi_{n-1}(\theta_2, ..., \theta_n)}\, \psi^s_n(\zeta, \theta_2, ..., \theta_n).$$

But since the integrand is integrable over $\mathcal{C}_0 \times \mathbb{R}^{n-1}$, we may reverse the order of the two integrals and conclude $\oint_{\mathcal{C}_0} d\zeta\, \psi^s_n(\zeta, \theta_2, ..., \theta_n) = 0$ for almost all $\theta_2, ..., \theta_n \in \mathbb{R}$, *i.e.* ψ^s_n is the boundary value of an analytic function in the first variable if the other variables $\theta_2, ..., \theta_n \in \mathbb{R}$ are held fixed.

Consider the symmetry condition (*cf.* (2.1))

$$\psi^s_n(\theta_2, \theta_1, \theta_3, ..., \theta_n) = S_2(\theta_1 - \theta_2) \cdot \psi^s_n(\theta_1, \theta_2, \theta_3, ..., \theta_n) \tag{4.45}$$

and let $\theta_2, ..., \theta_n \in \mathbb{R}$ be fixed. In view of the analyticity of the scattering function in the enlarged strip $S(-\kappa_g, \pi + \kappa_g)$, we see that the right hand side of (4.45) is the boundary value of a function analytic in its first variable θ_1, in the region $S(-\kappa_g, \pi + \kappa_g) \cap S(-\pi, 0) = S(-\kappa_g, 0)$. Hence the left hand side can be continued to $S(-\kappa_g, 0)$ as well, and we conclude that ψ^s_n has also an analytic extension in the second variable, to the strip $S(-\kappa_g, 0)$, if the other variables are held fixed. In the same way we see inductively from

$$\psi^s_n(\theta_1, ..., \theta_{k+1}, \theta_k, ..., \theta_n) = S_2(\theta_k - \theta_{k+1}) \psi^s_n(\theta_1, ..., \theta_k, \theta_{k+1}, ..., \theta_n) \tag{4.46}$$

that ψ^s_n is analytic in each variable $\theta_k \in S(-\kappa_g, 0)$ if the remaining $(n-1)$ variables are held fixed and real. (We neglect here the even stronger analyticity in the first variable in the region $S(-\pi, 0)$.) By application of the Malgrange-Zerner theorem (for a proof of this theorem see [18]) we conclude that ψ^s_n is analytic (as a function of n complex variables) in the tube region

$$\mathscr{T}_n(\kappa_g) := \mathbb{R}^n - i\mathscr{B}_n(\kappa_g), \tag{4.47}$$

$$\mathscr{B}_n(\kappa_g) := \left\{ \boldsymbol{\lambda} \in \,]\,0, \kappa_g\,[^{\times n} :\; 0 < \sum_{k=1}^n \lambda_k < \kappa_g \right\}. \tag{4.48}$$

To avoid the divergencies due to the poles of S_2 at the boundary of $\mathscr{T}_n(\kappa_g)$, we now fix some $\kappa \in \,]\,0, \kappa_g]$ and consider the smaller tube $\mathscr{T}_n(\kappa)$ instead of $\mathscr{T}_n(\kappa_g)$. Note that $\mathscr{B}_n(\kappa)$ contains the n-dimensional cube $]\,0, \frac{\kappa}{n}\,[^{\times n}$. As

$$(\Delta^\lambda \psi^s_n)(\boldsymbol{\theta}) = \psi^s_n(\theta_1 - 2\pi\lambda \cdot i, ..., \theta_n - 2\pi\lambda \cdot i)\,, \tag{4.49}$$

the analyticity of ψ^s_n in $\mathscr{T}_n(\kappa)$ implies the claim of Lemma 4.2 a), and the parameter α appearing there can be chosen as

$$\alpha = \frac{\kappa}{4\pi n}\,. \tag{4.50}$$

Now we use the uniform bound (4.41) to prove part b). The relations (4.45) and (4.44) imply $\|\psi_n^s(\,.\,,\,.-i\lambda_2,...)\| \leq \sigma(\lambda_2)b(\lambda_2,s)\|A\|$, and inductively we get from (4.46)

$$\int d^n\boldsymbol{\theta}\,|\psi_n^s(\theta_1,...,\theta_k-i\lambda_k,...,\theta_n)|^2 \;\;\leq\;\; \sigma(\lambda_k)^{2(k-1)}b(\lambda_k,s)^2\cdot\|A\|^2. \tag{4.51}$$

Thus the analytic continuations of ψ_n^s in each single variable have L^2-norm bounded by $\sigma(\kappa)^{n-1}b(\kappa,s)\|A\|$. This bound can be transported to the interiour of the tube $\mathscr{T}_n(\kappa)$ by using the flat tube theorem, see Lemma A.2 in the appendix. We arrive at

$$\int d^n\boldsymbol{\theta}\,|\psi_n^s(\boldsymbol{\theta}-i\boldsymbol{\lambda})|^2 \;\;\leq\;\; \sigma(\kappa)^{2(n-1)}b(\kappa,s)^2\cdot\|A\|^2\,, \quad \boldsymbol{\lambda}\in\overline{\mathscr{B}_n(\kappa)}\,. \tag{4.52}$$

In particular, the norm bound claimed in Lemma 4.2 b) follows, and the parameter K can be chosen as

$$K=\sigma(\kappa)^{n-1}\cdot b(\kappa,s)\,. \tag{4.53}$$

To establish the limit $\lim_{|\boldsymbol{\theta}|\to\infty}\psi_n^s(\boldsymbol{\theta}-i\boldsymbol{\lambda})=0$, let $\boldsymbol{\theta}\in\mathbb{R}^n$, $\boldsymbol{\lambda}\in\mathscr{B}_n(\kappa)$ and consider a polydisc $\mathscr{D}_\rho\subset\mathscr{T}_n(\kappa)$ with sufficiently small radius ρ and $\boldsymbol{\theta}-i\boldsymbol{\lambda}\in\mathscr{D}_\rho$. By the mean value property for analytic functions and Hölder's inequality,

$$\begin{aligned}|\psi_n^s(\boldsymbol{\theta}-i\boldsymbol{\lambda})|^2 \;&\leq\; (\pi\rho^2)^{-n}\int_{\mathscr{D}_\rho} d^n\boldsymbol{\zeta}'\,|\psi_n^s(\boldsymbol{\zeta}')|^2\\ &\leq\; (\pi\rho^2)^{-n}\int_{[-\rho,\rho]^{\times n}} d^n\boldsymbol{\lambda}' \int_{[-\rho,\rho]^{\times n}} d^n\boldsymbol{\theta}'|\psi_n^s(\boldsymbol{\theta}'+\boldsymbol{\theta}+i(\boldsymbol{\lambda}'-\boldsymbol{\lambda}))|^2\,.\end{aligned}$$

Because of (4.52) the last integral is convergent, also when the integration in $\boldsymbol{\theta}'$ is carried out over $\mathbb{R}^n$ instead of $[-\rho,\rho]^{\times n}$. Hence it vanishes in the limit $|\boldsymbol{\theta}|\to\infty$. Finally, c) is a consequence of the earlier discussed strong continuity (Lemma 4.4 d)) of $[0,\kappa]\ni\lambda_1\mapsto\psi_{n,\lambda_1}^s$. □

5 Summary and Outlook

We have reviewed a novel approach to the construction of quantum field theories with a factorizing S-matrix on two-dimensional Minkowski space. Starting from a scattering function describing the interaction of one type of massive, scalar particles without bound states, in a first step certain auxiliary quantum fields ("polarization-free generators") were constructed. It was shown how to define a covariant net of wedge algebras from these fields. Furthermore, we mentioned that the "correct" two-particle scattering behaviour, namely the one expected from the input scattering function, can be recovered from the wedge-local fields.

In a second step, the local operator content of these wedge local theories has to be analyzed to ensure physical meaningful models. In this context, the modular nuclearity condition constitutes a sufficient criterion for the existence of local observables. In two particular examples, namely the models with constant scattering functions ± 1, this criterion has been verified already.
In the present paper, the modular compactness criterion, a necessary condition for the split property of the wedge net and thus providing an intermediate step in proving the existence of local operators, has been checked in a wide class of models with typical, non-constant scattering functions. In view of these results, it seems reasonable to conjecture that the question of the existence of local observables will have an affirmative answer in the family of models considered.
If this conjecture can be proven, the program reviewed here provides a possibility to rigorously construct interacting quantum field theories in two dimensions, without taking recourse to classical concepts. Although the family of S-matrices considered is limited, it seems possible to generalize the procedure to more complicated models with several kinds of massive particles, ultimately leading to an existence proof for quantum field theories with arbitrary factorizing S-matrix.

Appendix

In this appendix we prove two Lemmata in complex analysis which are used in the main text. The first one is an adaptation of the three lines Theorem [31, Thm 12.8] to the case of L^2-bounds, and the second one shows how to obtain such bounds in the situation of the Malgrange-Zerner theorem. Both statements seem to be well-known; but as we did not find them in this form in the literature, we give their proofs here.

Lemma A.1 *Let $a, b \in \mathbb{R}$, $a < b$,*

$$S(a,b) \;\;:=\;\; \{z \in \mathbb{C} \,:\, a < \mathrm{Im}(z) < b\}, \tag{A.1}$$

and let f denote a function which is analytic in $S(a,b)$. Assume that for each $y \in [a,b]$, the function $x \mapsto f(x+iy) =: f_y(x)$ is an element of $L^2(\mathbb{R}, dx)$ and that $f_y \to f_c$ for $y \to c$, where $c = a, b$, in the sense of distributions.
Then the limits $f_y \to f_c$ are also valid in the norm topology of $L^2(\mathbb{R}, dx)$, and

$$\|f_y\| \;\;\leq\;\; \max\{\|f_a\|, \|f_b\|\}\,, \qquad a \leq y \leq b\,. \tag{A.2}$$

Proof: Let $\widetilde{g} \in C_0^\infty(\mathbb{R})$. Then g is entire analytic and $x \mapsto g(x+iy) =: g_y(x)$ is of rapid decrease at infinity for fixed $y \in \mathbb{R}$. Let $0 < y < b - a$.

$$\int dp\, \widetilde{f}_{a+y}(-p)\widetilde{g}(p) \;\;=\;\; \int dx\, f_{a+y}(x) g(x) = \int dx\, f_{a+\varepsilon y}(x) g_{(\varepsilon-1)y}(x).$$

Here we used the rapid decrease of g_y and the analyticity of f and g to shift the integration from $\mathbb{R}$ to $\mathbb{R} + iy(\varepsilon - 1)$, where $0 < \varepsilon < 1$. As $\widetilde{g} \in C_0^\infty(\mathbb{R})$, the limit

$\lim_{\varepsilon\to 0} g_{(\varepsilon-1)y} = g_{-y}$ holds in the topology of $\mathscr{S}(\mathbb{R})$. Together with the distributional convergence $f_{a+\varepsilon y} \to f_a$, this implies that the above integral is equal to

$$\int dx\, f_a(x) g(x - iy) \;\; = \;\; \int dp\, \widetilde{f}_a(-p) e^{yp} \widetilde{g}(p) \,.$$

Hence $\widetilde{f}_{a+y}(p) = e^{-yp}\widetilde{f}_a(p)$. This implies

$$\|f_{a+y} - f_a\|^2 \;\; = \;\; \int dp\, |\widetilde{f}_a(p)|^2 (e^{-yp} - 1)^2 \,,$$

and since we have the integrable bound

$$|\widetilde{f}_a(p)|^2 (e^{-yp} - 1)^2 \;\; \leq \;\; |\widetilde{f}_a(p)|^2 \left(4\Theta(p) + \Theta(-p)(1 + e^{-p(b-a)})^2\right) ,$$

we may use Lebesgue's dominated convergence theorem to conclude $\lim_{y\searrow a} f_y = f_a$ in the norm toplogy of $L^2(\mathbb{R}, dx)$. The limit $\lim_{y\nearrow b} f_y = f_b$ is established in the same manner.
Now let $h \in \mathscr{S}(\mathbb{R})$ be a test function and consider the convolution $f * h$, which is an analytic function in $S(a, b)$. It satisfies the bound, $a < y < b$,

$$|(f * h)(x + iy)| \;\; \leq \;\; \int dx'\, |h(x')| \cdot |f_y(x - x')| \leq \|h\| \cdot \|f_y\| < \infty \,.$$

But in view of the above established continuity of $[a, b] \ni y \mapsto f_y$, the norm $\|f_y\|$ depends continuously on y, and hence we can find a uniform bound on $|(f * h)(z)|$, $z \in \overline{S(a, b)}$. By the three lines theorem, we conclude

$$\left| \int dx'\, h(x') f_y(x - x') \right| \;\; \leq \;\; \|h\| \cdot \max\{\|f_a\|, \|f_b\|\}, \qquad a \leq y \leq b \,.$$

As $h \in \mathscr{S}(\mathbb{R})$ was arbitrary and $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R}, dx)$, the claim follows. □

Lemma A.2 *Let*

$$\mathscr{B}_n := \left\{ \mathbf{y} \in (0,1)^{\times n} \; : \; 0 < \sum_{j=1}^{n} y_j < 1 \right\}, \qquad \mathscr{T}_n := \mathbb{R}^n + i\mathscr{B}_n \tag{A.3}$$

and consider an analytic function $f : \mathscr{T}_n \to \mathbb{C}$ *of* n *complex variables. Setting* $f_{\mathbf{y}} : \mathbb{R}^n \to \mathbb{C}, \mathbf{x} \mapsto f(\mathbf{x} + i\mathbf{y})$*, assume that* $f_{\mathbf{y}} \in L^2(\mathbb{R}^n, d^n\mathbf{x})$ *for any* $\mathbf{y} \in \mathscr{B}_n$*, that the map* $\mathbf{y} \mapsto f_{\mathbf{y}}$ *can be extended norm continuously to* $\overline{\mathscr{T}_n}$*, and that*

$$\|f_{(0,\dots,y_k,\dots 0)}\|_2 \;\; \leq \;\; 1, \qquad 0 < y_k < 1,\; k \in \{1, \dots, n\}, \tag{A.4}$$

holds.
Then one has the bound

$$\|f_{\mathbf{y}}\|_2 \;\; \leq \;\; 1 \tag{A.5}$$

for any $\mathbf{y} \in \overline{\mathscr{B}_n}$*.*

Proof: Let $g \in \mathscr{S}(\mathbb{R}^n)$ be a test function and consider the convolution $f * g$. According to the hypothesis of the Lemma, $f*g$ is analytic in $\mathscr{T}_n$ and $\mathbf{y} \longmapsto (f*g)_{\mathbf{y}}$ is continuous on $\overline{\mathscr{T}_n}$. On the boundary we have

$$|(f*g)(x_1, ..., x_k + iy_k, ..., x_n)| \;\; \leq \;\; \|g\|_2 \|f_{(0,...,y_k,...0)}\|_2 \; \leq \; \|g\|_2. \qquad \text{(A.6)}$$

Now consider $h_g(\mathbf{z}) := ((f*g)(\mathbf{z}) - e^{i\alpha}(\|g\| + \varepsilon))^{-1}$, where $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ are arbitrary. Due to the bound (A.6), h_g is, in each variable seperately, analytic in the strip $S(0,1)$ if the remaining variables are held fixed and real. By the Malgrange-Zerner Theorem, we conclude that h_g has an analytic continuation, as a function of n complex variables, to the tube $\mathscr{T}_n$. Varying α and letting $\varepsilon \to 0$, we conclude

$$|(f*g)(\mathbf{x} + i\mathbf{y})| \;\; = \;\; |\langle \hat{g}_{\mathbf{x}}, \, f_{\mathbf{y}} \rangle| \; \leq \; \|g\|_2 \; = \; \|\hat{g}_{\mathbf{x}}\|_2 \,, \qquad \text{(A.7)}$$

where we have put $\hat{g}_{\mathbf{x}}(\mathbf{x}') := \overline{g(\mathbf{x} - \mathbf{x}')}$. But as $g \in \mathscr{S}(\mathbb{R}^n)$ was arbitrary and $\mathscr{S}(\mathbb{R}^n)$ is norm dense in $L^2(\mathbb{R}^n, d^n\mathbf{x})$, the claim (A.5) follows. □

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