Deformations of Operator Algebras and the Construction of Quantum Field Theories

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This note outlines a novel approach to the construction of quantum field theories on four-dimensional Minkowski space, which is based on operator-algebraic techniques. It is explained how Rieffel deformations can be used to deform quantum field theories in a manner which is compatible with Poincaré covariance and locality.

Keywords: Deformations of algebras, Rieffel deformation, deformation quantization, non-commutative geometry

1. Introduction: Wedge triples and local nets

In relativistic quantum field theory, the rigorous construction of interacting models in four space-time dimensions is still an open, challenging problem for mathematical physics. Despite the insights we have gained from constructive quantum field theory [1] and other methods [2], even the existence of interacting quantum field theories satisfying standard assumptions has not been proved until today, and it seems that new ideas are needed to improve this situation.

The aim of this contribution is to outline a recent development which uses operator-algebraic deformation techniques as a new tool in the construction of QFT models. This method is adapted to the framework of algebraic QFT [3, 4], where models are specified in terms of their local observable algebras: For each spacetime region $O \subset \mathbb{R}^4$ (in four-dimensional Minkowski space), one considers the algebra $\mathcal{A}(O)$ generated by all observables localized in O. Basic properties like Einstein causality and relativistic covariance can be formulated naturally in this algebraic language, and nowadays many tools exist to extract important physical quantities, such as the particle content, S-matrix, charge structure, thermal equilibrium states and local field content, from the *local net* $O \mapsto \mathcal{A}(O)$ [3] .

In all algebraic construction procedures developed so far, a particular kind of unbounded region in Minkowski space plays a special role, the so-called *right wedge*

$$W_R := \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \,:\, x_1 > |x_0|\}\,.$$

The following algebraic structure is associated with this region.

Definition 1.1. A *wedge triple* $(\mathcal{A}, \mathcal{B}, \alpha)$ consists of an inclusion $\mathcal{A} \subset \mathcal{B}$ of C^*-algebras and a strongly continuous automorphic action of the Poincaré group $\mathcal{P}$ on $\mathcal{B}$, such that

(1) $\alpha_g(\mathcal{A}) \subset \mathcal{A}$ for all $g \in \mathcal{P}$ with $gW_R \subset W_R$,
(2) $\alpha_{g'}(\mathcal{A}) \subset \mathcal{A}' \cap \mathcal{B}$ for all $g' \in \mathcal{P}$ with $g'W_R \subset -W_R$,

where $\mathcal{A}' \cap \mathcal{B}$ denotes the relative commutant of $\mathcal{A}$ in $\mathcal{B}$.

The significance of wedge triples for QFT relies on the following facts [5]. On the one hand, any QFT defines a wedge triple: Take $\mathcal{B}$ as the C^*-algebra of all observables of the theory, localized anywhere in Minkowski space, and $\mathcal{A}$ as the C^*-algebra of all observables localized in the wedge W_R. Since we are considering relativistic field theories, $\mathcal{B}$ carries an automorphic action of the Poincaré group, and the two *consistency conditions* in the above definition hold because of the locality and covariance properties of the theory.

On the other hand, any wedge triple gives rise to an associated QFT. Namely, starting from a triple $(\mathcal{A}, \mathcal{B}, \alpha)$, one can define the "wedge algebras"

$$\mathcal{A}(\Lambda W_R + x) := \alpha_{x,\Lambda}(\mathcal{A}) \,, \tag{1}$$

and the consistency conditions imply that this assignment of wedges (Poincaré transforms of W_R) to C^*-algebras, $W \mapsto \mathcal{A}(W)$, complies with the usual isotony, covariance and locality assumptions of QFT. Moreover, given a bounded spacetime region O, one can unambiguously construct the maximal algebra $\mathcal{A}(O)$ of all observables localized in O. These local algebras inherit the basic covariance and locality properties from the wedge triple. Since the construction of such $\mathcal{A}(O)$ involves intersections of subalgebras of $\mathcal{B}$, one has to check however that these algebras $\mathcal{A}(O)$ are non-trivial, i.e. that the QFT associated to the triple $(\mathcal{A}, \mathcal{B}, \alpha)$ contains strictly localized observables.

This two-sided relation between QFT models and wedge triples opens up the possibility to realize new QFTs by exploring examples of wedge triples. The construction then consists of first finding an appropriate triple $(\mathcal{A}, \mathcal{B}, \alpha)$, and then working out the physics of the QFT associated to this triple.

As explained in [5], for the construction of QFTs in their vacuum representation, one can work with a particular concrete form of wedge triples: Consider a Fock space $\mathcal{H}$ with its second quantized unitary strongly continuous representation U of $\mathcal{P}$, and let $\mathcal{B} := \mathcal{B}(\mathcal{H})$, $\alpha := \mathrm{ad}\, U$. Then the task is to find $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ such that the two conditions of Def. 1.1 are satisfied. In this setting, the structure of $\mathcal{H}$ and U is dictated by the particle spectrum of the QFT to be described, and the choice of $\mathcal{A}$ encodes the interaction.

In two dimensions, the constructive program making use of wedge triples has already been carried through for a large family of interacting models, where the algebra $\mathcal{A}$ is set up with the help of a factorizing S-matrix in the spirit of inverse scattering theory [6–9]. Also in the four-dimensional case, a number of authors have used wedge-localized objects for constructive purposes [10–13]. First wedge triples corresponding to interacting QFTs have been constructed in [14] and then generalized in [5] and [15]. These examples will be reviewed from a novel point of view in the following.

As non-trivial wedge triples are hard to construct from scratch, we will in the following consider the question how wedge triples can be *deformed*. That is, we will assume a triple $T := (\mathcal{A}, \mathcal{B}, \alpha)$ complying with Def. 1.1 to be given (for example by an interaction-free theory) and then "perturb" this structure in the family of wedge triples to some T_θ. Here θ is a "coupling constant" such that $T_0 = T$, and the QFT corresponding to the deformed triple T_θ exhibits non-trivial interaction for $\theta \neq 0$.

2. Rieffel deformations of wedge triples

The particular deformation of wedge triples which we want to discuss here relies on Rieffel' work on deformations of C^*-algebras, and we briefly recall some aspects of this analysis [16]. Rieffel considers a unital C^*-algebra $\mathcal{B}$ equipped with a strongly continuous action α of $\mathbb{R}^d$ by automorphisms. In our application to wedge triples, $\mathcal{B}$ will be the larger algebra of a wedge triple $(\mathcal{A}, \mathcal{B}, \alpha)$, and the action α is given by restricting the Poincaré action of this triple to the translation subgroup $\mathbb{R}^4$.

Rieffel's analysis is based on a deformed product $\times_\theta$ which can be introduced on the dense subalgebra $\mathcal{B}^\infty \subset \mathcal{B}$ of elements $B \in \mathcal{B}$ for which $x \mapsto \alpha_x(B)$ is smooth. To define it in our context, we equip $\mathbb{R}^4$ with the Minkowski inner product and consider as deformation parameter a real (4×4)-matrix θ which is antisymmetric with respect to the Minkowski product. Then the deformed product $\times_\theta$ on $\mathcal{B}^\infty$ is given by the integral formula

$$A \times_\theta B := (2\pi)^{-d} \int_{\mathbb{R}^d} dp \int_{\mathbb{R}^d} dx \, e^{-ipx} \, \alpha_{\theta p}(A) \alpha_x(B) \, , \qquad A, B \in \mathcal{B}^\infty \, . \tag{2}$$

Note that the integral (2) has to be defined in an oscillatory sense.

Some results established in the general context of Rieffel deformations are the following: $\times_\theta$ carries $\mathcal{B}^\infty$ into itself and is jointly continuous in the natural Fréchet topology of this algebra, it is an associative product and reproduces the undeformed product in $\mathcal{B}$ for $\theta = 0$, i.e. $A \times_0 B = AB$. Furthermore, $\times_\theta$ is compatible with the identity and star involution of $\mathcal{B}$.

We now consider a wedge triple $(\mathcal{A}, \mathcal{B}, \alpha)$, and want to use Rieffel's procedure to deform it. For the purposes of this note, we restrict ourselves to consider only a realization associated with a $\mathcal{P}$-invariant state. That is, in the following we put $\mathcal{B} := \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ which carries a strongly continuous unitary representation U of the Poincaré group with a U-invariant vector Ω. The action α is the adjoint action of U on $\mathcal{B}(\mathcal{H})$, and $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ a C^*-subalgebra.

To deform $\mathcal{A}$, we consider the left multipliers $B \mapsto (A \times_\theta B)$ w.r.t. the deformed product (2). Since Ω is U-invariant, these products evaluate on Ω to

$$A_\theta \;:\; B\Omega \longmapsto (A \times_\theta B)\Omega = (2\pi)^{-4} \int dp \int dx\, e^{-ipx}\, U(\theta p) A U(x - \theta p)\, B\Omega\,, \quad (3)$$

and define a new, deformed, operator A_θ, if the integral is interpreted in the same oscillatory sense as before. This operator can also be represented as a "warped convolution" of A by the spectral measure of U [5], and this point of view shows in particular that the p-integration in (3) runs only over the spectrum $S \subset \mathbb{R}^4$ of U, while the x-integration runs over all of $\mathbb{R}^4$. That is, we have

$$A_\theta = (2\pi)^{-4} \int_S dp \int dx\, e^{-ipx}\, U(\theta p) A U(x - \theta p)\,, \quad (4)$$

and define the deformed wedge algebra $\mathcal{A}_\theta$ as the C^*-algebra generated by all A_θ, where $A \in \mathcal{A}$ is smooth.

We thus consider the deformed triple $(\mathcal{A}_\theta, \mathcal{B}(\mathcal{H}), \mathrm{ad}\, U)$, and turn to the crucial question under which conditions these data still define a wedge triple, such that it can be used to build a QFT model.

Because of the form of the conditions in Def. 1.1, one has to study the effect of Lorentz transformations on the operators A_θ to answer this question. A proof cannot be given in this short contribution, but we will at least sketch what the main mechanisms are which are relevant for making the Rieffel deformation preserve the wedge triple structure.

One first computes that general Poincaré transformations (a, Λ) consisting of a translation $a \in \mathbb{R}^4$ and a Lorentz transformation Λ act on the Rieffel product $\times_\theta$ according to

$$\alpha_{a,\Lambda}(A \times_\theta B) = \alpha_{a,\Lambda}(A) \times_{\Lambda\theta\Lambda^{-1}} \alpha_{a,\Lambda}(B)\,. \quad (5)$$

We thus have to compare Rieffel deformations with different deformation parameters, and first choose a suitable θ adapted to the geometry of W_R. Let

$$\theta := \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_2 \\ 0 & 0 & -\kappa_2 & 0 \end{pmatrix}, \qquad \kappa_1, \kappa_2 \in \mathbb{R}\,, \quad (6)$$

which is antisymmetric w.r.t. the Minkowski inner product. Then the transformations (a, Λ) appearing in Def. 1.1 can be characterized as follows [14]:

- $\Lambda W_R + a \subset W_R \;\Leftrightarrow \Lambda\theta\Lambda^{-1} = \theta$ and $a \in \overline{W_R}$,
- $\Lambda W_R + a \subset -W_R \;\Leftrightarrow \Lambda\theta\Lambda^{-1} = -\theta$ and $a \in -\overline{W_R}$.

So the wedge-preserving transformations appearing in the first condition in Def. 1.1 preserve θ, and the wedge-reflecting transformations appearing in the second condition map θ to $-\theta$.

With this choice of θ, it is straightforward to verify that the deformed algebra $\mathcal{A}_\theta$ satisfies the first consistency condition for any $\kappa_1, \kappa_2 \in \mathbb{R}$.

The second condition, related to the locality of the deformed theory, is more involved. Since the transformations mapping W_R into its causal complement correspond to $-\theta$, we must study the relation between the Rieffel deformations with parameters θ and $-\theta$. The second condition is valid if for all smooth $A \in \mathcal{A}, A' \in \mathcal{A}', B \in \mathcal{B}(\mathcal{H})$,

$$0 = A \times_\theta (A' \times_{-\theta} B) - A' \times_{-\theta} (A \times_\theta B) = (A_\theta A'_{-\theta} - A'_{-\theta} A_\theta) B\Omega\,.$$

Using Rieffel's formula (2), this commutator can be computed as the integral

$$[A_\theta,\, A'_{-\theta}] = (2\pi)^{-4} \int_S dp \int dx\, e^{-ipx} U(\tfrac{x}{2})[\alpha_{\theta p}(A),\, \alpha_{-\theta p}(A')]U(\tfrac{x}{2})\,, \tag{7}$$

where as before, S denotes the joint spectrum of the generators of the representation of the translations.

Since $\mathcal{A}$ is stable under translations in the direction of $\overline{W_R}$, the commutant $\mathcal{A}'$ is stable under translations in the opposite direction. Hence if $\theta p \in \overline{W_R}$ for all p in the above integral, the commutator vanishes. As observed by Buchholz and Summers [5], this situation is realized if one considers a vacuum representation where S is contained in the forward lightcone, and the parameter κ_1 is non-negative.

Theorem 2.1. *Let θ be of the form (6), and let the spectrum of $U|_{\mathbb{R}^4}$ be contained in the forward lightcone. Then the Rieffel-deformed triple $(\mathcal{A}_\theta, \mathcal{B}(\mathcal{H}), \mathrm{ad}U)$ is a wedge triple for any $\kappa_1 \geq 0$, $\kappa_2 \in \mathbb{R}$.*

This theorem has been obtained in the context of warped convolution deformations [5]. If it is applied to an interaction-free field theory, this wedge triple defines a new QFT model [14].

The discussion of the properties of this field theory goes beyond the scope of this note. We only mention here that the two-particle S-matrix changes under the deformation [5, 14], which proves in particular that the deformed wedge triple is not equivalent to the undeformed one. In application to Wightman quantum field theories, the deformed theory is governed by θ-dependent fields ($\tilde{\phi}^\theta(p) = \tilde{\phi}(p)U(\theta p)$ in the scalar case), and can thus be interpreted as a field theory on non-commutative Minkowski space [15]. This point of view also explains why the QFTs obtained by Rieffel deformations have strong non-local features and do not contain strictly localized observables [5].

The details of the construction outlined here will soon be published in a joint paper with D. Buchholz and S. J. Summers. Beyond the example of Rieffel deformations, an infinite class of other deformations of wedge triples has been found which give rise to similar field theories[a]. It seems that the family of wedge triples which lead to new QFT models is very large, and it would be interesting to systematically investigate which models can be obtained from a free field theory by such deformation procedures.

[a] G. Lechner, work in progress

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